101. A Remark on the S-Equation for Branching Processes

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Let $\{x_t, \mathscr{B}_t\}$ be a right-continuous strong Markov process on a metric space D. Assume $\{\phi_t\}$ is a finite increasing additive functional of $\{x_t\}$, and let $\{\pi_n(a, E)\}$ be a series of substochastic kernels for $a \in D$, $E \subseteq D^n$ satisfying $\sum \pi_n(a, D^n) = 1$.1) Consider the branching process $\{z_t\}$ in D (actually in $X = \bigcup_0^\infty D^n$) determined by $\{x_t\}$, the branching rate $d\phi_t$ $\left(\text{thus if } \phi_t = \int_0^t V(x_s)d_s\text{, an individual particle branches with probability } V(x_t)dt \text{ in time } dt\right)$ and position distributions $\pi_n(x_t, E)$ of the offspring of a particle which does branch. (See [1]-[5]; we use the notation of [4].) The transition function $\bar{P}(t, x, E)$ of $\{z_t\}$ in X can be determined from the transition function $P(t, a, A)$ of $\{x_t\}$ by either a linear equation in X or a non-linear equation in D. The linear equation is the so-called "M-equation".

$$(1)\qquad \bar{T}_t h(x) = E_x(h(z_t)\chi_{[\beta>t]}) + E_x\left(\chi_{[\beta\le t]}\int_X \bar{T}_{t-\beta}h(y)\mu(w, dy)\right)$$

for bounded Borel functions $h(x)$ on X, where $\bar{T}_t h(x) = \int h(y)\bar{P}(t, x, dy)$ and β is the first branching time ($P_a(\beta > t/\mathscr{B}_\infty) = \exp(-\phi_t)$ in D). Alternately, for $a \in D$, we have the "S-equation" ([5])

$$(2)\qquad \bar{T}_t \hat{f}(a) = E_a(f(x_t)\chi_{[\beta>t]}) + E_a\left(\chi_{[\beta\le t]}\sum_0^\infty \int_{D^r} \prod_1^r \bar{T}_{t-\beta}\hat{f}(b_i)\pi_r(x_\beta, db)\right)$$

where $f(a)$ is a Borel function on D, $|f(a)| \le 1$, and $\hat{f}(x) = \prod_1^r f(a_i)$ for $x = (a_1, a_2, \cdots, a_r)$, $f(\partial) = 1$. In particular, if new particles are always born at the same location where their parent branches, the non-linearity in (2) is of power series type. As is proven in [2, III], the semi-group $\{\bar{T}_t\}$ can be obtained from either equation.2) I.e., if $h(x) \ge 0$ (or $f(a) \ge 0$), then $\bar{T}_t h(x)$ (or $\bar{T}_t \hat{f}(a)$) is the minimal non-

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1) Here D^n is the usual n-fold Cartesian product of D with itself, and $D^0 = \{\partial\}$, $\partial \notin D$, where $\pi_0(a, \{\partial\}) = \pi_0(a)$ refers to x_t dying childless.

2) More exactly, in the case of (2), only those properties of the model which are permutation invariant; see the remark.

negative solution of (1) or (2) and can be obtained by iteration.

A probabilistic interpretation of iteration in (1) is easily found. If $\{\beta_n\}$ are the consecutive branching times of the process $\{z_t\}$, and $\bar{T}_t^{(n)}h(x)$ is the n-th iterate in equation (1) (with $\bar{T}_t^{(0)}h(x)=0$), then

$$\bar{T}_t^{(n)}h(x)=\bar{E}_x[h(z_t)\chi_{[\beta_n>t]}].$$

Or, if $g_t(w)$ is a random variable giving the total number of births (or deaths) before time t,

$$\bar{T}_t^{(n)}h(x)=\bar{E}_x[h(z_t)\chi_{[g_t\leq n-1]}].$$

The purpose here is to give a similar interpretation of iteration in (2). We assign "generation numbers" to all extant (or dead) particles, which begin with $n=0$ for the initial particles and increase by one in all branches from parent to each offspring, the other particles being unaffected. Let $l_t(w)$ be the *maximum* generation number of all particles (including extinguished ones) at time t. Thus $l_t(w)\leq 5$ iff all particles can trace their ancestry back to an original particle in five generations or less, and no particle has died childless in the fifth generation. Define

$$\phi_n(x,t)=\bar{E}_x(\hat{f}(z_t)\chi_{[l_t\leq n-1]})$$
$$\phi_n(\partial,t)=1$$

for all $x\in X$. Then, by arguing as in the proof of Lemma 6.1 in [4]

Lemma. *For all* $n\geq 1$, $a\in D$, and $x=(a_1,a_2,\cdots,a_r)\in D^r$,

$$\phi_n(a,t)=E_b(f(x_t)\chi_{[\beta>t]}) + E_a\left(\chi_{[\beta\leq t]}\sum_0^\infty\int_{D^r}\phi_{n-1}(y,t-\beta)\pi_r(x_\beta,dy)\right) \tag{3}$$

$$\phi_n(x,t)=\prod_1^r\phi_n(a_i,t). \tag{4}$$

By combining (3) and (4) we obtain the following

Theorem. *For any Borel function* $f(a)$ *on* D, $|f(a)|\leq 1$, *let* $\bar{T}_t^{(n)}\hat{f}(a)$ *be the n-th iterate in the non-linear equation* (2) *beginning with* $\bar{T}_t^{(0)}\hat{f}(a)=0$. *Then*

$$\bar{T}_t^{(n)}\hat{f}(a)=\bar{E}_a(\hat{f}(z_t)\chi_{[l_t\leq n-1]}) =\bar{E}_a(\prod f(x_t^{(i)}\chi_{\mathscr{B}_n})$$

where the product is over the (random) number of particles alive at time t and $\mathscr{B}_n$ *is the event* $l_t\leq n-1$.

Remark. The terms D^n of X in [4] (and here) are the usual Cartesian product and not the *symmetrized* Cartesian product of [2]; i.e., in [2], n-tuples which are permutations of one another are identified. The chief reason for the construction in [4] was for a simpler state space and construction of the process, although the unsymmetrized model does contain extra information. For example, the heir and heir apparent of a single initially existing particle would be instantly recognizable from $z_t(w)$, assuming that in any branch the first born particle always goes to the first component of the batch of components which replaces the parent. For a more detailed analysis of the "descendence-structure" of $\{z_t\}$, at least when $A=0$, see [6].

References

[1] N. Ikeda, M. Nagasawa, and S. Watanabe: Proc. Japan Acad., **41**, 816-821 (1965), and **42**, 252-257, 370-375, 380-384, 719-724, 1016-1021, and 1021-1026 (1966).
[2] ——: Branching Markov Processes. I-III. Jour. Math. Kyoto Univ., **8**, 233-278, 365-410 (1968); III (to appear).
[3] K. Ito and H. P. McKean, Jr.: Diffusion processes and their sample paths. Springer, Berlin, 1965, Chapter 5.
[4] S. Sawyer: A formula for semi-groups, with an application to branching diffusion processes (to appear).
[5] V. Skorokhod: Branching diffusion processes. Theor. Prob. Appl., **9**, 492-497 (1964) (English translation).
[6] V. P. Kharlamov: On properties of branching processes with an arbitrary set of particle types. Theor. Prob. Appl., **13**, 84-98 (1968) (English translation).