118. *On a Theorem of E. Michael and K. Nagami*

By Masao SAKAI

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E. Michael [1] and K. Nagami [2] proved the following theorem.

Theorem. *Every metacompact and collectionwise normal space is paracompact.*

A space is called metacompact if every open covering of it can be refined by a point-finite open covering.

We shall prove this theorem, using the following lemmas for point-finite open covering of a topological space.

Lemma 1. *Every point-finite open covering* $\{U_\alpha\}_{\alpha\in A}$ *of a topological space contains an irreducible subcovering* (see [3]).

Lemma 2. *For each point-finite open covering* $\{U_\alpha\}_{\alpha\in A}$ *of a normal space, there exists an open covering* $\{V_\alpha\}_{\alpha\in A}$ *of the space such that* $\bar{V}_\alpha\subset U_\alpha$ *for every* $\alpha\in A$ (see [4]).

Proof of Theorem. 1. Let X be a meta-compact and collectionwise normal space, and $\boldsymbol{C}$ be an any open covering of X. Then, there exists a point-finite open refinement $\boldsymbol{U}=\{U_\alpha\}_{\alpha\in A}$ of the covering $\boldsymbol{C}$. By Lemma 1, we can assume that the covering $\boldsymbol{U}$ is irreducible. We are going to prove that the open covering $\boldsymbol{U}$ has a σ-discrete open refinement $\{\boldsymbol{W}_n\}_{n=1,2,3,\cdots}$, $\boldsymbol{W}_n=\{W_{n,\beta}\}_{\beta\in B_n}$. Then, the space X will be paracompact.

2. By Lemma 2, for the irreducible open covering $\{U_\alpha\}_{\alpha\in A}$, there exists an open refinement $\{V_\alpha\}_{\alpha\in A}$ such that $\bar{V}_\alpha\subset U_\alpha$, for every $\alpha\in A$. *Put* $F_\alpha=\bar{V}_\alpha-\bigcup_{\substack{\alpha'\in A\\ \alpha'\neq\alpha}}U_{\alpha'}$, *then* $F_\alpha\neq\phi$, *for every* $\alpha\in A$.

For, if $F_\alpha=\phi$, then $\bigcup_{\substack{\alpha'\in A\\ \alpha'\neq\alpha}}U_{\alpha'}\supset\bar{V}_\alpha\supset V_\alpha$. By $U_{\alpha'}\supset V_{\alpha'}$,

$$\bigcup_{\substack{\alpha'\in A\\ \alpha'\neq\alpha}}U_{\alpha'}\supset V_\alpha\cup\bigcup_{\substack{\alpha'\in A\\ \alpha'\neq\alpha}}V_{\alpha'}=X.$$

This contradicts to the irreducibility of covering $\{U_\alpha\}_{\alpha\in A}$. Then $F_\alpha\neq\phi$.

3. *Any point x of the space X which is contained in only one* U_α *of the family* $\{U_\alpha\}_{\alpha\in A}$, *is contained in* F_α.

For, suppose that $x\in U_\alpha$ and $x\notin U_{\alpha'}$ $(\alpha'\in A,\alpha'\neq\alpha)$. By $\bar{V}_{\alpha'}\subset U_{\alpha'}$, $x\notin V_{\alpha'}$. As the family $\{V_\alpha\}_{\alpha\in A}$ is a covering of X, $x\in V_\alpha\subset\bar{V}_\alpha$. Then, $x\in\bar{V}_\alpha-\bigcup_{\substack{\alpha'\in A\\ \alpha'\neq\alpha}}U_{\alpha'}=F_\alpha$.

4. *The family* $\{F_\alpha\}_{\alpha\in A}$ *is a closed discrete family.*

For, any point x of X is contained in some U_α. For a neighborhood of x, put $V(x)=U_\alpha$.

By $U_\alpha \supset \bar{V}_\alpha \supset F_\alpha$,

$$V(x) \cap F_\alpha = U_\alpha \cap F_\alpha = F_\alpha \neq \phi.$$

For $\alpha' \in A, \alpha' \neq \alpha$,

$$V(x) \cap F_{\alpha'} = U_\alpha \cap (\bar{V}_{\alpha'} - \bigcup_{\substack{\alpha'' \in A \\ \alpha'' \neq \alpha'}} U_{\alpha''}) \subset U_\alpha \cap (X - U_\alpha) = \phi.$$

From these, the family $\{F_\alpha\}_{\alpha \in A}$ is discrete. That the sets $F_\alpha (\alpha \in A)$ are closed is obvious.

5. Being X collectionwise normal, there exists an open discrete family $\{G_\alpha\}_{\alpha \in A}$ such that $G_\alpha \supset F_\alpha$ for every $\alpha \in A$. Put $W_\alpha = G_\alpha \cap U_\alpha$ for every $\alpha \in A$. The following is obvious.

$F_\alpha \subset W_\alpha \subset U_\alpha$ and the family $\mathbf{W}_1 = \{W_\alpha\}_{\alpha \in A}$ is open discrete.

6. Put $W_1 = \bigcup_{\alpha \in A} W_\alpha$. By 3, the following is obvious. *Any point x of the space X which is contained in only one U_α of the family $\{U_\alpha\}_{\alpha \in A}$, is contained in W_1.*

7. Put $X_2 = X - W_1$. Being W_1 open, X_2 is closed. As the space X is normal, the subspace X_2 is normal. Then, we can apply Lemmas 1 and 2 in the subspace X_2.

By 6, any point of X_2 is contained in above two sets U_α of the family $\{U_\alpha\}_{\alpha \in A}$. Then, $X_2 \subset \bigcup_{\substack{\alpha \in A \\ \alpha' \in A \\ \alpha \neq \alpha'}} U_\alpha \cap U_{\alpha'}$. From this, the family $\{X_2 \cap U_\alpha \cap U_{\alpha'}\}_{\substack{\alpha \in A \\ \alpha' \in A \\ \alpha \neq \alpha'}}$ is a point-finite open covering of the subspace X_2. For this covering, there exists an irreducible open subcovering $\{U_{2,\beta}\}_{\beta \in B_2}$.

8. By Lemma 2, for the open covering $\{U_{2,\beta}\}_{\beta \in B_2}$ there exists an open refinement $\{V_{2,\beta}\}_{\beta \in B_2}$ such that $\bar{V}_{2,\beta} \subset U_{2,\beta}$ for every $\beta \in B_2$. Put $F_{2,\beta} = \bar{V}_{2,\beta} - \bigcup_{\substack{\beta' \in B_2 \\ \beta' \neq \beta}} U_{2,\beta'}$. Then, the family $\{F_{2,\beta}\}_{\beta \in B_2}$ is closed discrete relative to X_2.

The family $\{F_{2,\beta}\}_{\beta \in B_2}$ is closed discrete relative to X. For, as X_2 is closed in X, the sets $F_{2,\beta}$ are closed in X, for every $\beta \in B_2$. By 6, for every point x of the subspaces X_2, there exists a neighborhood $U_{2,\beta}$ $(\beta \in B_2)$ such that $U_{2,\beta}$ meets only one $F_{2,\beta'}$ $(\beta' \in B_2)$. By 7, there exist some U_α and $U_{\alpha'}$ $(\alpha, \alpha' \in A)$ such that $U_{2,\beta} = X_2 \cap U_\alpha \cap U_{\alpha'}$. For a neighborhood of x relative to X, put $V(x) = U_\alpha \cap U_{\alpha'}$. Then, for any $\beta' \in B_2$

$$V(x) \cap F_{2,\beta'} = U_\alpha \cap U_{\alpha'} \cap X_2 \cap F_{2,\beta'} = U_{2,\beta} \cap F_{2,\beta'}.$$

For $\beta' \neq \beta$ $(\beta' = \beta)$, $V(x) \cap F_{2,\beta'} = \phi$ $(\neq \phi)$. Then, the family $\{F_{2,\beta}\}_{\beta \in B_2}$ is closed discrete.

9. Being the space X collectionwise normal, there exists an open discrete family $\{G_{2,\beta}\}_{\beta \in B_2}$ such that $G_{2,\beta} \supset F_{2,\beta}$. For $U_{2,\beta}$ there exists $\alpha, \alpha' \in A, \alpha \neq \alpha'$ such that $U_{2,\beta} = X_2 \cap U_\alpha \cap U_{\alpha'}$. Put $W_{2,\beta} = G_{2,\beta} \cap U_\alpha \cap U_{\alpha'}$. The following is obvious.

The open family $\mathbf{W}_2 = \{W_{2,\beta}\}_{\beta \in B_2}$ is discrete.

10. Put $W_2 = \bigcup_{\beta \in B_2} W_{2,\beta}$. As in 3 and 6, we have the followings.

Any point of X_2 which is contained in only one set $U_{2,\beta}$ of the family $\{U_{2,\beta}\}_{\beta\in B_2}$ is contained in W_2.

Any point of X which is contained in just two sets U_α of the family $\{U_\alpha\}_{\alpha\in A}$, is contained in $W_1\cup W_2$.

11. Put $X_3=X_2-W_2=X-(W_1\cup W_2)$. As in 7, we have the following: any point of X_3 is contained in above 3 sets U_α of the family $\{U_\alpha\}_{\alpha\in A}$. And we have an open discrete family $\boldsymbol{W}_3=\{W_{3,\beta}\}_{\beta\in B_3}$.

12. By induction for $n=1, 2, 3, \cdots$, we have open discrete families $\boldsymbol{W}_n=\{W_{n,\beta}\}_{\beta\in B_n}$, where $\{W_{1,\beta}\}_{\beta\in B_1}$ is $\{W_\alpha\}_{\alpha\in A}$. Put $W_n=\bigcup_{\beta\in B_n}W_{n,\beta}$. Any point of X which is contained in just n sets U_α of the family $\{U_\alpha\}_{\alpha\in A}$, is contained in $\bigcup_{k=1}^{n}W_k$. Then $X\subset\bigcup_{n=1}^{\infty}\bigcup_{\beta\in B_n}W_{n,\beta}$, and the families $\{\boldsymbol{W}_n\}_n=1,2,3,\cdots$ are σ-open discrete refinement of $\{U_\alpha\}_{\alpha\in A}$, that is, σ-open discrete refinement of the covering $\boldsymbol{C}$. Then the space X is paracompact.

References

[1] E. Michael: Point-finite and locally-finite converings. Canad. Jour. of Math., **7**, 275–279 (1955).

[2] K. Nagami: Paracompactness and strong screenability. Nagoya Math. Jour., **8**, 83–88 (1955).

[3] R. Arens and J. Dugundji: Remark on the concept of compactness. Portugaliae Math., **9**, 141–143 (1950).

[4] S. Lefschetz: Algebraic Topology. New York (1942).