193. *On Closed Graph Theorem*

By Michiko NAKAMURA

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1970)

This paper is to give a type of closed graph theorem for topological linear spaces similar to the one discussed in the previous paper [4], generalizing and simplifying the results obtained in [1], [2], and [3].

We make use of the notations in [4].

A filter Φ in a linear space E is said to be a *LS-filter* if Φ is generated by the complements of all the finite union of linear subspaces E_n $(n=1,2,\cdots)$ such that $E=\bigcup_{n=1}^{\infty} E_n$.

A subset A of a linear space E is said to be *linearly* open if for any straight line L in E, $L\cap A$ is *open* in L by its usual topology.

A filter Φ in a linear space E is said to be a *P-filter* if for every x in E there exists a linearly open set A such that either A is disjoint from Φ or Φ_A, considered as a filter in E, is finer than a LS-filter. (In general, we identify a filter Ψ in a subset of E with a filter in E generated by Ψ.)

A linear topological space E (in the sequel we suppose that every linear topological space is Hausdorff) is called a *generalized netted space* (called GN-space in the sequel) if there exists a sequence of P-filters Φ_n $(n=1,2,\cdots)$ such that every ultrafilter Ψ with $\Psi \not\succ \Phi_n$ $(n=1, 2,\cdots)$ converges in E.

E is called a *pre-GN-space* if there exists a sequence of P-filters Φ_n $(n=1,2,\cdots)$ such that every ultrafilter Ψ with $\Psi \not\succ \Phi_n$ $(n=1,2,\cdots)$ is a Cauchy-filter in E. The P-filters Φ_n, in these cases, are called defining filters for E.

Let φ be a linear mapping from a linear space E into a linear space F. The image $\varphi(A)$ of a linearly open subset A of E by φ is linearly open in $\varphi(E)$ and the inverse image $\varphi^{-1}(B)$ of a linearly open subset B in $\varphi(E)$ by φ is linearly open in E.

If φ is an one-to-one linear mapping from E into F, then the image $\varphi(\Phi)$ of a P-filter Φ in E is a P-filter.

If φ is a linear mapping from E into F, then the inverse image $\varphi^{-1}(\Phi)$ of a P-filter Φ in F such that $\varphi(E)$ is not disjoint from Φ is a P-filter in E. In particular, if E is a linear subspace of F, then for every P-filter Φ in F such that E is not disjoint from Φ, Φ_E is a P-filter in E, and a P-filter in E can be considered as a P-filter in F.

By virtue of these facts, we can see easily that the class of GN-

spaces, as in the case of quasi-Souslin spaces (in [4]), is closed by the following operations:

(1) *The image* $E=\varphi(F)$ *of a GN-space* F *by a continuous linear mapping* φ *is a GN-space.*

(2) *The closed subspace* E *of a GN-space* F *is a GN-space.*

(3) *The product space* $E=\prod_n E_n$ *of GN-spaces* E_n $(n=1,2,\cdots)$ *is a GN-space.*

(4) *The inductive limit* E *of GN-spaces* E_n $(n=1,2,\cdots)$ *is a GN-space.*

First we prove that every metric linear space E is a pre-GN-space. Let d be the distance function in E. Put $U_n(x)=\left\{y\in E\,|\,d(x,y)<\frac{1}{n}\right\}$. Let Φ_n be the filter generated by the complements of all the finite union of $U_n(x)$ for all x in E. Then Φ_n is obviously a P-filter in E and we show that E is a pre-GN-space with the defining filters Φ_n $(n=1,2,\cdots)$. Let Ψ be an ultrafilter in E such that $\Psi \not\supset \Phi_n$ for every n, then there exists a sequence of elements x_n $(n=1,2,\cdots)$ in E such that $\Psi\ni U_n(x_n)$. Then Ψ is a Cauchy-filter in E.

Proposition. *Let* E *be a linear topological space with a system of subspaces* $E_{n_1,n_2,\cdots,n_k}$ *defined for every finite sequence* $n_1,n_2,\cdots,n_k$ *of natural numbers such that*

$$E=\bigcup_{n=1}^{\infty}E_n,\ E_{n_1}=\bigcup_{n=1}^{\infty}E_{n_1,n},\ \cdots,$$

$$E_{n_1,n_2,\cdots,n_k}=\bigcup_{n=1}^{\infty}E_{n_1,n_2,\cdots,n_k,n},\ \cdots.$$

Suppose a topology of pre-GN-space is given on each $E_{n_1,n_2,\cdots,n_k}$ *such that for every infinite sequence* $\{n_k\}_{k=1,2,\cdots}$, *if the restriction of a filter* Ψ *in* $E_{n_1,n_2,\cdots,n_k}$ *is a Cauchy-filter for every* $k=1,2,\cdots$, *then* Ψ *converges in* E. *Then* E *is a GN-space.*

Proof. For each $\{n_1,n_2,\cdots n_k\}$, let $\Phi^i_{n_1,n_2,\cdots,n_k}$ $(i=1,2,\cdots)$ be defining P-filters for $E_{n_1,n_2,\cdots,n_k}$, $\Psi_{n_1,n_2,\cdots,n_k}$ the filter generated by the complements in $E_{n_1,n_2,\cdots,n_k}$ of all the finite union of $E_{n_1,n_2,\cdots,n_k,n}$ $(n=1,2,\cdots)$, and Ψ_0 the filter generated by the complement in E of all the finite union of E_n $(n=1,2,\cdots)$. We will prove that E is a GN-space with the defining filters $\Phi^i_{n_1,n_2,\cdots,n_k}$, $\Psi_{n_1,n_2,\cdots,n_k}$ $(i,n_1,n_2,\cdots,n_k)$ and Ψ_0, these filters being consider as P-filters in E. Let Ψ be an ultrafilter in E such that $\Psi\not\supset\Phi^i_{n_1,n_2,\cdots,n_k}$, $\Psi\not\supset\Psi_{n_1,n_2,\cdots,n_k}$ for every $i,n_1,n_2,\cdots,n_k$ and $\Psi\not\supset\Psi_0$. From $\Psi\not\supset\Psi_0$ there exists a natural number n_1 such that $\Psi\ni E_{n_1}$ and from $\Psi\not\supset\Psi_{n_1}$ there exists a nutural number n_2 such that $\Psi\ni E_{n_1,n_2}$. Continuing this process, we can find a sequence $\{n_k\}_{k=1,2\cdots}$ such that every $E_{n_1,n_2,\cdots,n_k}$ $(k=1,2,\cdots)$ belongs to Ψ. Since $\Psi\not\supset\Phi^i_{n_1,n_2,\cdots,n_k}$ for every $i,k=1,2,\cdots$, the restriction of Ψ in $E_{n_1,n_2,\cdots,n_k}$ is a Cauchy-filter for every $k=1,2,\cdots$, and hence, by the assumption, Ψ converges in E. Thus E is a GN-space.

Netted spaces of [1] and [2], and $\alpha\beta\gamma$-representable spaces of [3] are GN-spaces, because they are special cases of the linear topological space E with a system of subspaces $E_{n_1,n_2,\cdots,n_k}$ satisfying the condition of the proposition, where the topology given on each $E_{n_1,n_2,\cdots,n_k}$ is metric.

Corresponding to the closed graph theorem for quasi-Souslin space in [4], we obtain the following

Theorem. *Every graph closed linear mapping φ from a linear topological space F of second category into a GN-space E is continuous.*

Proof. Let E be a GN-space with defining P-filters Φ_n. If we can prove that there exists a sequence of subsets A_i $(i=1,2,\cdots)$ of F such that A_i is everywhere second category in F, $A_i\supset A_{i+1}$, and, for every x in A_i, there exists a neighborhood U of x such that $U\cap A_i$ is disjoint from $\varphi^{-1}(\Phi_i)$, then the same argument as in the corresponding proof in [4] can be applied to complete our proof. We put $F=A_0$. Now we show that we obtain a sequence of subsets A_i $(i=1,2,\cdots)$ such that each A_i satisfies the following condition (*) in addition to the above condition.

(*) *For every x in A_i there exist a linearly open set $L\ni x$ in F and a second category linear subspace $H\ni x$ of F such that $A_i\supset L\cap H$.*

Clearly A_0 satisfies the condition (*). For each i when A_i is already determined, we determine A_{i+1} in the following way. Let $\{(V_\lambda,B_\lambda)\}_{\lambda\in\Lambda}$ be a maximal family of pairs (V_λ,B_λ) with the following conditions (1) to (4):

(1) B_λ is everywhere second category in non-void open set V_λ.

(2) $A_i\supset B_\lambda$, B_λ is disjoint from $\varphi^{-1}(\Phi_{i+1})$.

(3) B_λ satisfies the condition (*).

(4) $V_\lambda\cap V_{\lambda'}=\emptyset$ if $\lambda\neq\lambda'$.

Put $A_{i+1}=\bigcup_{\lambda\in\Lambda}B_\lambda$. Suppose there exists an open set W such that $W\cap V_\lambda=\emptyset$ for all $\lambda\in\Lambda$. For x in $W\cap A_i$, there exist a linearly open set $L\ni x$ in F and a second category linear subspace $H\ni x$ of F such that $W\cap A_i\supset L\cap H\ni x$. As Φ_{i+1} is a P-filter in E, $\varphi^{-1}(\Phi_{i+1})$ is also a P-filter in F. So there exists a linearly open set $M\ni x$ in F such that M is either disjoint from $\varphi^{-1}(\Phi_{i+1})$ or $\{\varphi^{-1}(\Phi_{i+1})\}_M$ is finer than a LS-filter Ψ defined by a sequence of subspaces F_k $(k=1,2,\cdots)$. Put $C=L\cap H\cap M$ in the first case, and, in the second case, $C=L\cap H\cap M\cap F_k$ where k is chosen as to let C be second category. Putting $V=O(C)$ and $B=V\cap C$, we obtain a pair (V,B) and the family $\{(V_\lambda,B_\lambda),(V,B)\}$ satisfying (1), (2), (3), and (4), contradicting the maximality of $\{(V_\lambda,B_\lambda)\}_{\lambda\in\Lambda}$. So we have proved that $\bigcup_{\lambda \in \Lambda} V_\lambda$ is dense in F, and then it is obvious that A_{i+1} satisfies all the required conditions.

References

[1] H. G. Garnir: Some new results in classical functional analysis. Proceedings of the International Conference on Functional Analysis and Related Topics, 361–368 (1970).

[2] M. De Wilde: Sur le théorème du graphe fermé. C. R. Acad. Sc. Paris, **265**, série A, 376–379 (1967).

[3] W. Slowikowski: On continuity of inverse operators. Bull. Amer. Math. Soc., **67** (5), 467–470 (1961).

[4] M. Nakamura: On quasi-Souslin space and closed graph theorem. Proc. Japan Acad., **46** (6), 514–517 (1970).