207. *On Dirichlet Series whose Coefficients are Class Numbers of Integral Binary Cubic Forms*

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1. In this note we give a concrete example of "zeta functions associated with prehomogeneous vector spaces" introduced by Professor M. Sato.

2. We denote by V the vector space of real binary cubic forms. For every $x=(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4$, we define $F_x \in V$ as follows:

$$F_x(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3.$$

In the following, we identify V with $\boldsymbol{R}^4$ by the linear isomorphism: $x \to F_x$. V becomes a $GL(2, \boldsymbol{R})$-module if we put

$$g \cdot F_x((u, v))=F_x((u, v)g)=F_{g\cdot x}((u, v)) \quad (g \in GL(2, \boldsymbol{R})).$$

For every $x \in V$, we denote by $P(x)$ the discriminant of F_x. We have

$$P(x)=x_2^2x_3^2+18x_1x_2x_3x_4-4x_1x_3^3-4x_2^3x_4-27x_1^2x_4^2$$

and

$$P(g\cdot x)=(\det g)^6P(x) \qquad (g \in GL(2, \boldsymbol{R})).$$

In the following, we put

$$\chi(g)=(\det g)^6 \qquad (g \in GL(2, \boldsymbol{R})).$$

For every $x, y \in V$, we put

$$\langle x, y\rangle=x_4y_1-\frac{1}{3}x_3y_2+\frac{1}{3}x_2y_3-x_1y_4.$$

We denote by $\mathcal{S}(V)$ the space of rapidly decreasing functions on V and define the Fourier transform $\hat{f}$ of $f \in \mathcal{S}(V)$ as follows:

$$\hat{f}(x)=\int_V e^{2\pi i\langle x, y\rangle}f(y)dy.$$

3. We denote by L the lattice of integral binary cubic forms. We have

$$L=\{F_x\,;\, x \in \boldsymbol{Z}^4\}.$$

Then L is invariant under the action of the $SL(2, \boldsymbol{Z})$. Two elements x, y of L are said to be equivalent if there exists a $\gamma \in SL(2, \boldsymbol{Z})$ such that $x=\gamma\cdot y$.

For every integer $m\neq 0$, we denote by L_m the set of integral binary cubic forms whose discriminants are m. It is known that there exist only finite number of equivalence classes in L_m. We denote by $h(m)$ the number of equivalence classes in L_m. Let

$$x_1, \cdots, x_{h(m)}$$

be the representatives of equivalence classes in L_m. When $m<0$, it is known that the isotropy subgroup of each x_i in $SL(2, \boldsymbol{Z})$ is $\{1\}$ $(1 \leqslant i \leqslant h(m))$. When $m>0$, the isotropy subgroup of each x_i in $SL(2, \boldsymbol{Z})$ is either $\{1\}$ or a cyclic group of order 3. In the first case we call x_i belongs to the class of the first kind. In the second case we call x_i belongs to the class of the second kind.

We denote by $h_1(m), h_2(m)$ the numbers of classes of the first and the second kind respectively. We put

$$\hat{L}=\{F(u, v)=x_1u^3+x_2u^2v+x_3uv^2+x_4v^3 \in L\,;\ x_2, x_3 \in 3\boldsymbol{Z}\}.$$

Then $\hat{L}$ is an $SL(2, \boldsymbol{Z})$-submodule of L. We denote by $\hat{h}(m)$ the number of equivalence classes in L_m which are contained in $\hat{L}$. We define $\hat{h}_1(m), \hat{h}_2(m)$ $(m>0)$ in a similar fashion. Now we define four Dirichlet series as follows:

$$\xi_1(L, s)=\sum_{n=1}^{\infty}\frac{h_1(n)+\frac{1}{3}h_2(n)}{n^s},$$

$$\xi_2(L, s)=\sum_{n=1}^{\infty}\frac{h(-n)}{n^s},$$

$$\xi_1(\hat{L}, s)=\sum_{n=1}^{\infty}\frac{\hat{h}_1(n)+\frac{1}{3}\hat{h}_2(n)}{n^s},$$

$$\xi_2(\hat{L}, s)=\sum_{n=1}^{\infty}\frac{\hat{h}(-n)}{n^s}.$$

4. We define a Haar measure dg on $GL(2, \boldsymbol{R})$ as follows:

$$dg=|\det g|^{-2}\, dp\, dq\, dr\, ds \qquad \left(g=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right).$$

We put

$$G_+=\{g \in GL(2, \boldsymbol{R})\,;\ \det g>0\},$$
$$\Gamma=SL(2, \boldsymbol{Z}),$$
$$L'=\{x \in L\,;\ P(x)\neq 0\}$$

and

$$\hat{L}'=\hat{L} \cap L'.$$

For every $f \in \mathcal{S}(V)$, we put

$$Z(f, L\,;\, s)=\int_{G_+/\Gamma}\chi(g)^s \sum_{x \in L'} f(g\cdot x)dg$$

and

$$Z(f, \hat{L}\,;\, s)=\int_{G_+/\Gamma}\chi(g)^s \sum_{x \in \hat{L}'} f(g\cdot x)dg \qquad (s \in \boldsymbol{C}).$$

Further we put

$$\Phi_1(f, s)=\int_{P(x)>0}|P(x)|^s f(x)dx$$

and put

$$\Phi_2(f, s)=\int_{P(x)<0}|P(x)|^s f(x)dx.$$

Then we have the following:

Proposition 1. (i) *When $Re(s)$ is sufficiently large, we have*

$$Z(f, L; s)=\xi_1(L, s)\Phi_1(f, s-1)+\frac{1}{3}\xi_2(L, s)\Phi_2(f, s-1)$$

and

$$Z(f, \hat{L}, s)=\xi_1(\hat{L}, s)\Phi_1(f, s-1)+\frac{1}{3}\xi_2(\hat{L}, s)\Phi_2(f, s-1).$$

(ii) *$Z(f, L, s)$ and $Z(f, \hat{L}, s)$ can be continued analytically as meromorphic functions of s in the whole plane and satisfy the following functional equation.*

$$Z(f, L, s)=Z(\hat{f}, \hat{L}, 1-s).$$

Proposition 2. *$\Phi_1(f, s)$ and $\Phi_2(f, s)$ are meromorphic functions of s in the whole plane and satisfy the following functional equation:*

$$\begin{pmatrix}\Phi_1(\hat{f}, s-1)\\ \Phi_2(\hat{f}, s-1)\end{pmatrix}=\Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)\frac{\pi^{-4s}3^{6s}}{18}\cdot\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin \pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\Phi_1(f, -s)\\ \Phi_2(f, -s)\end{pmatrix}.$$

Using these two propositions we can prove the following theorem.

Theorem. (i) *Four Dirichlet series defined above converge absolutely when $Re\ s>1$ and can be continued analytically as meromorphic functions in the whole plane which have simple poles at $s=1$ and $s=5/6$. They satisfy the following functional equation*

$$\begin{pmatrix}\xi_1(L, 1-s)\\ \xi_2(L, 1-s)\end{pmatrix}=\Gamma\left(s-\frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s+\frac{1}{6}\right)\frac{\pi^{-4s}3^{6s}}{18}\cdot\begin{pmatrix}\sin 2\pi s & \sin \pi s\\ 3\sin \pi s & \sin 2\pi s\end{pmatrix}\begin{pmatrix}\xi_1(\hat{L}, s)\\ \xi_2(\hat{L}, s)\end{pmatrix}$$

(ii) *Residues of them at $s=1$ and at $s=5/6$ are given in the following table.*

Table of Residues

at \ of	$\xi_1(L, s)$	$\xi_2(L, s)$	$\xi_1(\hat{L}, s)$	$\xi_2(\hat{L}, s)$
$s=1$	$\frac{\pi^2}{9}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{162}$	$\frac{\pi^2}{81}$
$s=\frac{5}{6}$	$\frac{\sqrt{3}}{18}r$	$\frac{1}{6}r$	$\frac{\sqrt{3}}{162}r$	$\frac{1}{54}r$

$$\left(\text{We put } r=\zeta\left(\frac{2}{3}\right)\frac{\Gamma(\frac{1}{3})(2\pi)^{1/3}}{\Gamma(\frac{2}{3})}.\right)$$

Detailed proof will appear elsewhere.

Reference

Mikio Sato: Theory of prehomogeneous vector spaces. Sûgaku no Ayumi, 15-1, pp. 85–157 (in Japanese).