8. A Note on Knots and Flows on 3-manifolds

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H. Seifert shows in [1] (Satz 11) that for any torus knot \( k \) in the 3-sphere \( S^3 \) there is a flow on \( S^3 \) with \( k \) as an orbit, and conversely, that if a homotopy 3-sphere \( \Sigma^3 \) admits a flow on it so that all orbits are closed then \( \Sigma^3 = S^3 \) and each orbit is a torus knot.

Here, we consider the following question: For any knot \( k \) in \( S^3 \) does there exist a non-singular flow on \( S^3 \) having \( k \) as an orbit, allowing for the flow having non-closed orbits? In this paper, we give an affirmative answer to this question.

Manifolds and maps, etc in this paper are assumed to be smooth \((C^\infty)\) ones. A flow on a manifold \( M \) is a 1-parameter group of transformations \( \phi: \mathbb{R} \times M \to M \) \((\mathbb{R}, \text{the real numbers})\). \( x \in M \) is said to be a singular point if \( \phi(t, x) = x \) for all \( t \in \mathbb{R} \). \( \phi \) is said to be non-singular if there is no singular point. An orbit of \( \phi \) passing \( x \) is a subset \( \{\phi(t, x) | t \in \mathbb{R}\} \). If there is \( t \neq 0 \) such that \( \phi(t, x) = x \), the orbit is said to be closed.

Let \( f \) be a map of \( S^1 \) into a space \( M \) and \( p: \mathbb{R} \to S^1 \) be the usual universal covering defined by \( t \to e^{2\pi i t} \), then we shall denote \( f \circ p = f \).

**Theorem.** Let \( M \) be an orientable closed 3-manifold and \( f: S^1 \to M \) be an embedding. Then, there exist a flow \( \phi: \mathbb{R} \times M \to M \) and \( x \in M \) such that \( \phi(t, x) = f(t) \) for all \( t \in \mathbb{R} \).

**Proof.** Denote the tangent bundle of \( M \) by \( T(M) \). Since, by [2] (Satz 21), \( M \) is parallelizable, we may assume \( T(M) = M \times \mathbb{R}^3 \). Consider the \((\mathbb{R} \setminus \{0\})\)-bundle \( T(M) \), \( \xi: M \times (\mathbb{R}^3 \setminus \{0\}) \to M \) over \( M \) associated to tangent bundle. We define a map \( g: f(S^1) \to T(M) \) as follows: for \( x \in f(S^1) \), \( g(x) = \frac{d}{dt}(f(t)) \) where \( t \) is any number such that \( f(t) = x \). \( g \) is well-defined. Since \( f \) is an embedding, \( g \) is a cross-section of \( \xi \) over \( f(S^1) \). We will extend \( g \) to a cross-section of \( \xi \) over \( M \).

We may take a tubular neighborhood \( U \) of \( f(S^1) \) coordinated as follows:

\[ U = \{(x, r, \theta) | x \in f(S^1), \ 0 \leq r \leq 1, \ 0 \leq \theta < 2\pi\} \]

with

\[ (x, 0, \theta) = (x, 0, 0) \text{ for all } x \text{ and } \theta. \]

Since \( \pi((\mathbb{R}^3 \setminus \{0\})) \cong \pi(S^3) = 0 \), we have a homotopy \( F \) of \( q \circ g \) as follows, where \( q \) is the projection into the second factor \( M \times (\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}^3 \setminus \{0\} : 

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Next, we define a map $G : M \to \mathbb{R}^3 - \{0\}$, as follows.

\[
G(x, r, \theta) = \begin{cases} 
q \circ g(x) & \text{if } 0 \leq r < \frac{1}{2} \\
F(x, 2r - 1) & \text{if } \frac{1}{2} \leq r \leq 1 \\
* & \text{if } y \in U.
\end{cases}
\]

$G$ is continuous. By an approximation keeping fixed on $f(S^1)$, we may make $G$ a smooth map $\tilde{G}$. If we put $(y, \tilde{G}(y)) = \tilde{g}(y)$, $\tilde{g} : M \to \mathbb{R}^3 - \{0\}$ is a cross-section of $\xi$, and also, it is an extension of $g$.

We may assume that $\tilde{g}$ is a non-zero vector field on $M$ extending $g$. The flow, obtained by integrating $\tilde{g}$, is the desired one. This proves the Theorem.

Let $l$ be an embedding \( \{ S_1^1 \cup \cdots \cup S_n^1 \} \to M \), where $S_1^1$ is a circle and $S_1^1 \cup \cdots \cup S_n^1$ is the disjoint union, then we call $l$ a link in $M$ and each $l(S_i)$ a component of the link.

**Corollary.** For any link $l$ of an orientable closed 3-manifold $M$, there exists a non-singular flow $\phi$ of $M$ such that each component of $l$ coincides with a certain orbit of $\phi$.

The proof is similar to the one of the Theorem.

**Remark.** There is a well-known Seifert’s Conjecture which states that every non-singular flow on $S^3$ has a closed orbit. The Theorem states that if we solve the Seifert’s Conjecture we must take it into consideration that any knot may come out as the closed orbit.

**References**
