63. On a Property of Behavior in Time for Solutions of the Wave Equation

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(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1971)

In this note we treat the solutions of the wave equation with even space dimension. In the case of odd space dimension, it is easily verified that the solutions of the wave equation with initial data in $\mathcal{S}$ (=the totality of Schwartz's rapidly decreasing functions) decrease rapidly when t tends to infinity. On the other hand, in the case of even dimension, this is not always true. Generally the solutions can only decay with t^{-N}. For this reason we argue whether there are the solutions which decay rapidly when t tends to infinity. The similar problems for the solutions of the second order hyperbolic equations are treated by many authors. See [1], [2], [3], [4], [5].

The author wishes to thank Mr. K. Hayakawa and Dr. K. Masuda who kindly suggested the improvement of the proofs.

We consider the following Cauchy problem:

$$(1)\qquad \frac{\partial^2 u}{\partial t^2}=\Delta u, \qquad \begin{pmatrix}\Delta=n\text{-dimensional}\\ \text{Laplacian};\ n=2p\end{pmatrix}$$

$$(2)\qquad u(x,0)=\varphi(x)\in\mathcal{S},$$

$$(3)\qquad u_t(x,0)=\psi(x)\in\mathcal{S}.$$

As is well known, the above Cauchy problem has the following unique solution;

$$(4)\qquad \begin{aligned} u(x,t)=&(2\pi)^{-p}\frac{d}{dt}\left(\frac{1}{t}\frac{d}{dt}\right)^{p-1}\left[t^{n-1}\int_{|\xi|\le 1}\frac{\varphi(x-t\xi)}{\sqrt{1-|\xi|^2}}d\xi\right]\\ &+(2\pi)^{-p}\left(\frac{1}{t}\frac{d}{dt}\right)^{p-1}\left[t^{n-1}\int_{|\xi|\le 1}\frac{\psi(x-t\xi)}{\sqrt{1-|\xi|^2}}d\xi\right].\end{aligned}$$

Theorem 1. *We fix an arbitrary x. For $u(x,t)$ decreases rapidly when t tends to infinity, it is necessary and sufficient that the following* (5) *and* (6) *are satisfied.*

$$(5)\qquad \int|x-\xi|^{2m}\,\varphi(\xi)d\xi=0, \qquad (m=0,1,2,\cdots),$$

$$(6)\qquad \int|x-\xi|^{2m}\,\psi(\xi)d\xi=0, \qquad (m=0,1,2,\cdots).$$

Proof. We put

$$J(t;\varphi)=t^{n-1}\int_{|\xi|\le 1}\frac{\varphi(x-t\xi)}{\sqrt{1-|\xi|^2}}d\xi=t^{-1}\int_{|\xi|\le 1}\frac{\varphi(x-\xi)}{\sqrt{1-|\xi|^2/t^2}}d\xi,$$

$$J_1(t\,;\varphi)=t^{-1}\int_{|\xi|\le t/2}\frac{\varphi(x-\xi)}{\sqrt{1-|\xi|^2/t^2}}d\xi,$$
$$J_2(t\,;\varphi)=t^{-1}\int_{t/2<|\xi|\le t}\frac{\varphi(x-\xi)}{\sqrt{1-|\xi|^2/t^2}}d\xi.$$

We can easily verify that $J_2(t\,;\phi)$ decreases rapidly. We consider $J_1(t\,;\phi)$.

We note the expansion formula of $(1-\tau^2)^{-1/2}$ for small τ;

$$(1-\tau^2)^{-1/2}=\sum_{j=0}^{N}C_j\tau^{2j}+\frac{1}{N\,!}\int_0^{\tau^2}(\tau^2-t)^N f^{(N+1)}(t)dt,\qquad (7)$$
$$f(t)=(1-t)^{-1/2}.$$

Substituting (7) in $J_1(t\,;\phi)$, we have

$$J_1(t\,;\varphi)=t^{-1}\sum_{j=0}^{N}C_jt^{-2j}\int_{|\xi|\le t/2}|\xi|^{2j}\varphi(x-\xi)d\xi+O(t^{-2N-3}).$$

Since

$$\int_{|\xi|\le t/2}|\xi|^{2j}\varphi(x-\xi)d\xi=\int|\xi|^{2j}\varphi(x-\xi)d\xi-\int_{|\xi|>t/2}|\xi|^{2j}\varphi(x-\xi)d\xi$$

and the second term decreases rapidly, finally we get

$$J_1(t\,;\xi)=t^{-1}\sum_{j=0}^{N}C_jt^{-2j}\int|\xi|^{2j}\varphi(x-\xi)d\xi+O(t^{-2N-3}).\qquad (8)$$

This formula is valid for any φ from $\mathcal{S}$ and for any N.

Now we are in a position to prove the Theorem. Substituting (8) in (4) and calculating the differentiation, we have the following

$$\begin{aligned}u(x,t)=&\sum_{j=0}^{N}C'_j(-2j-2p+1)t^{-2j-2p}\int|\xi|^{2j}\varphi(x-\xi)d\xi\\&+\sum_{j=0}^{N}C'_jt^{-2j-2p+1}\int|\xi|^{2j}\psi(x-\xi)d\xi+O(t^{-2N-2p-1})\end{aligned}\qquad (9)$$

where C'_j's are non zero and independent of N. When we multiply the both sides of (9) by t^{2p-1} and let t go to infinity, the left hand side tends to zero and the right hand side tends to $C'_0\int\psi(x-\xi)d\xi$. Consequently we get

$$\int\psi(x-\xi)d\xi=0.$$

From this fact, it follows that the second term of the right hand side of (9) is the order of t^{-2p}. Noting this fact we multiply the both sides of (9) by t^{2p} again and let t go to infinity, then we get

$$\int\varphi(x-\xi)d\xi=0.$$

Repeating this argument, we finally get

$$\int|\xi|^{2m}\varphi(x-\xi)d\xi=0,$$
$$\int|\xi|^{2m}\psi(x-\xi)d\xi=0.$$

Conversely, if (5) and (6) are satisfied, from (9) $u(x,t)=O(t^{-m})$ for any non-negative integer m. This completes the proof.

Lemma. *Let φ satisfy* (5). *φ is identically zero if there are positive numbers C and ε such that*

$$|\varphi(x)| \leqq C e^{-\varepsilon|x|}. \tag{10}$$

Proof. Considering the right hand side of (5) as a polynomial of x, we get

$$\int \xi^{\alpha} \varphi(\xi) d\xi = 0 \tag{11}$$

for any multi-index α.

Since the series

$$\sum_{j=0}^{\infty} C_j (x \cdot \xi)^j \varphi(\xi), \qquad C_j = \frac{(-i)^j}{j\,!}$$

uniformly converges to $e^{-ix\cdot\xi}\varphi(\xi)$ in R_n if $|x|<\varepsilon$, it follows from (11) that

$$O = \sum_{j=0}^{\infty} C_j \int (x\cdot\xi)^j \varphi(\xi) d\xi = \int e^{-ix\cdot\xi}\varphi(\xi) d\xi, \qquad (|x|<\varepsilon). \tag{12}$$

On the other hand $\int e^{-ix\cdot\xi}\varphi(\xi)d\xi$ is an analytic regular function of several complex variables in $|\mathcal{I}m x|<\varepsilon$. From this fact and (11) we can use the theorem of identity and then we get

$$\int e^{-ix\cdot\xi}\varphi(\xi)d\xi = 0, \qquad \text{for any } x \in R_n,$$

therefore we have $\varphi \equiv 0$. q.e.d.

We get easily the following Theorem 2 from Theorem 1 and lemma.

Theorem 2. *Let $u(x, t)$ be the solution of the Cauchy problem* (1), (2) *and* (3) *where φ and ψ satisfy* (10). *If $u(x, t)$ decreases rapidly when t tends to infinity, $u(x, t)$ is identically zero.*

Remark 1. In case $\varphi, \psi \in \mathcal{D}$, $u(x, t)$ represented as a power series of t^{-1} for sufficiently large t;

$$u(x, t) = \sum_{j=0}^{\infty} (-2j-2p+1) C'_j t^{-2j-2p} \int |\xi|^{2j} \varphi(x-\xi) d\xi$$
$$+ \sum_{j=0} C'_j t^{-2j-2p+1} \int |\xi|^{2j} \psi(x-\xi) d\xi.$$

Remark 2. Lemma is not always true for any φ in $\mathcal{S}$. Choose a function f in $\mathcal{S}$ such that it vanishes in a neighborhood of origin and let φ be Fourier transformation of f. If we operate the well known formula

$$f(x)e^{-ix\cdot y} = (2\pi)^{-n} \int e^{ix\cdot(\xi-y)} \varphi(\xi) d\xi, \qquad \xi = (2\pi)^{-n} d\xi$$

by $(\Delta_x)^m$, we have

$$(\Delta_x)^m [f(x)e^{-ix\cdot y}] = (2\pi)^{-n} \int (-|y-\xi|^2)^m e^{ix\cdot(\xi-y)} \varphi(\xi) d\xi.$$

If we put $x=0$, we get

$$\int |\xi - y|^{2m} \varphi(\xi) d\xi = 0, \qquad \text{for any } y \text{ in } R_n.$$

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