62. On the Asymptotic Distribution of Eigenvalues of Operators Associated with Strongly Elliptic Sesquilinear Forms

By Kenji MARUO and Hiroki TANABE

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1971)

1. Introduction and main theorem. The object of this note is to show that concerning the asymptotic distribution of eigenvalues of elliptic operators the results similar to those of S. Agmon [1], [2], R. Beals [3], etc. hold under somewhat different assumptions. Only an outline of the proof is presented here and the details will be published elsewhere.

Let Ω be a bounded domain of R^n having the restricted cone property ([2]). Let V be a closed subspace of $H_m(\Omega)$ containing $\mathring{H}_m(\Omega)$ and $a(u, v)$ be a symmetric integro-differential sesquilinear form of order m:

$$a(u, v)=\int_\Omega \sum_{|\alpha|, |\beta|\leq m} a_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}\,dx.$$

It is assumed that there exists a positive constant δ such that

$$a(u, u)\geqq \delta \|u\|_m^2 \qquad \text{for any } u\in V.$$

It is also assumed that $2m>n$. We denote by V^* the antidual of V. Then according to the usual convention we may consider $V\subset L^2(\Omega)\subset V^*$ algebraically and topologically. Let A be the operator associated with the sesquilinear form a:

$$a(u, v)=(Au, v) \qquad \text{for } u, v\in V,$$

where the bracket on the right denotes the pairing between V^* and V. A is a bounded linear operator on V onto V^*. For $x\in\Omega$ let $\delta(x)=\min\{1, \operatorname{dist}(x, \partial\Omega)\}$. We denote by $N(t)$ the number of eigenvalues of A which do not exceed $t>0$.

Theorem. *Suppose that the coefficients of the highest order terms of a are Hoelder continuous of order h and other coefficients are bounded and measurable. Suppose also that*

$$\int_\Omega \delta(x)^{-p}dx<\infty$$

for some positive number $p<1$. *Under the hypotheses stated above we have*

(1) $$N(t)=c_0 t^{n/2m}+O(t^{(n-\theta)/2m})$$

as $t\to\infty$ *where*

---

*) Part of the contents of this paper was presented at the Conference on Evolution Equations and Functional Analysis, University of Kansas, Lawrence, Kansas, June-July, 1970.

$$c_0 = \frac{\sin(n\pi/2m)}{n\pi/2m} \int_\Omega c_0(x)dx,$$

$$c_0(x) = (2\pi)^{-n} \int_{R^n} \{ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta + 1\}^{-1} d\xi,$$

and θ is an arbitrary positive number smaller than $h/(h+3)$.

Furthermore if $a_{\alpha\beta}$, $|\alpha|=|\beta|=m$, are functions of class $C^{2+h}(\Omega_1)$ where Ω_1 is a domain containing $\bar{\Omega}$, and $a_{\alpha\beta}$, $|\alpha|+|\beta|=2m-1$, are of class $C^{1+h}(\Omega_1)$, and $a_{\alpha\beta}$, $|a|+|\beta|=2m-2$, are of class $C^h(\Omega_1)$, then (1) *holds for any $\theta \in (0, (h+2)/(h+5))$.*

Remark. In this theorem it is assumed that $2m>n$; however, the domain of A considered as a closed operator in $L^2(\Omega)$ need not be contained in $H_{2m}(\Omega)$. The existence of such an example is shown by the following observation. Letting (x, y) be the generic point of R_2 we consider the function $u=r^{3/2}\sin(3\theta/2)=\mathrm{Im}\,(x+iy)^{3/2}$. In the upper half plane $y>0$ $\Delta u=0$ and hence $\Delta^2 u=0$. For $x>0, y=0$ $u=\partial^2 u/\partial y^2=0$, and for $x<0, y=0$ $\partial u/\partial y=\partial^3 u/\partial y^3=0$. Near the orgin $u \notin H_3$ although $u \in H_2$ there.

2. Outline of the proof of the main theorem.

Lemma 1. *Let S be a bounded linear operator on V^* to V, then S has a kernel M in the following sense:*

$$(Sf)(x) = \int_\Omega M(x, y) f(y) dy \qquad \text{for } f \in L^2(\Omega).$$

There exists a constant C such that

$$|M(x, y)| \leqq C\, \|S\|_{V^*\to V}^{n^2/4m^2} \|S\|_{V^*\to L^2}^{n/2m-n^2/4m^2} \|S\|_{L^2\to V}^{n/2m-n^2/4m^2} \|S\|_{L^2\to L^2}^{(1-n/2m)^2}$$

for any $x, y \in \Omega$. Here $\|S\|_{V^*\to V}$ denotes the norm of S considered as an operator on V^* to V and similarly for other norms.

Proof. Applying Sobolev's inequality as a function of y

$$|M(x, y)| \leqq \gamma\, \|M(x, \cdot)\|_m^{n/2m} \|M(x, \cdot)\|_0^{1-n/2m}.$$

Taking into account that $L^2(\Omega)$ is dense in V^* we have

$$\|M(x, \cdot)\|_m = \sup_{f\in L^2} \left| \int M(x, y) f(y) dy \right| / \|f\|_{V^*}$$

$$= \sup_{f\in L^2} |(Sf)(x)| / \|f\|_{V^*}.$$

Again by Sobolev's inequality

$$|(Sf)(x)| \leqq \gamma\, \|Sf\|_m^{n/2m} \|Sf\|_0^{1-n/2m} \leqq \gamma\, \|S\|_{V^*\to V}^{n/2m} \|S\|_{V^*\to L^2}^{1-n/2m} \|f\|_{V^*}.$$

Hence

$$\|M(x, \cdot)\|_m \leqq \gamma\, \|S\|_{V^*\to V}^{n/2m} \|S\|_{V^*\to L^2}^{1-n/2m}.$$

$\|M(x, \cdot)\|_0$ can be estimated in a similar manner and combining these inequalities we obtain the lemma.

For a complex number λ let $d(\lambda)$ be the distance from λ to the positive real axis.

Lemma 2. *There exists a constant C such that*

$$\|(A-\lambda)^{-1}\|_{V^*\to V} \leqq C\, |\lambda| / d(\lambda),$$

$$\|(A-\lambda)^{-1}\|_{V^*\to L^2} \leqq C\, |\lambda|^{1/2} / d(\lambda),$$

$$\|(A-\lambda)^{-1}\|_{L^2\to V^*}\leqq C|\lambda|^{1/2}/d(\lambda),$$
$$\|(A-\lambda)^{-1}\|_{L^2\to L^2}\leqq d(\lambda)^{-1}.$$

Let A_1 be the operator associated with the restriction of a to $\mathring{H}_m(\Omega)\times\mathring{H}_m(\Omega)$:

$$a(u,v)=(A_1u,v)\ \text{ for }\ u,v\in\mathring{H}_m(\Omega).$$

A_1 is a bounded operator on $\mathring{H}_m(\Omega)$ onto the antidual $H_{-m}(\Omega)$ of $\mathring{H}_m(\Omega)$. The inequalities similar to the ones stated in Lemma 2 hold for A_1. Let K_λ and K_λ^1 be the kernels of $(A-\lambda)^{-1}$ and $(A_1-\lambda)^{-1}$ respectively.

Lemma 3. *For any $p\geqq 0$ the following inequality holds*:

$$|K_\lambda(x,x)-K_\lambda^1(x,x)|\leqq C\frac{|\lambda|^{n/2m}}{d(\lambda)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda)}\right)^p,\ |\lambda|\geqq 1,$$

where C is a constant depending on p but not on x and λ.

This lemma can be proved applying Lemma 1 to the operator

$$Sf=\zeta((A-\lambda)^{-1}f-(A_1-\lambda)^{-1}(rf))$$

where ζ is a smooth function with a small support near x and rf is the restriction of $f\in V^*$ to $\mathring{H}_m(\Omega)$.

Lemma 4. *Under the present assumptions the following inequality holds for any $p\geqq 0$*:

$$|K_\lambda^1(x,x)-c_0(x)(-\lambda)^{n/2m-1}|$$
$$\leqq C\frac{|\lambda|^{n/2m}}{d(\lambda)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda)}\right)^p+C|\lambda|^{(n-\theta)/2m-1}$$

for $|\lambda|\geqq 1$, $d(\lambda)\geqq|\lambda|^{1-\theta/2m}$, where $c_0(x)$ and θ are the same ones defined in the main theorem and C is a constant depending on p but not on x and λ.

Combining the above lemmas and Malliavin's tauberian theorem we obtain the main theorem.

References

[1] S. Agmon: Asymptotic formulas with remainder estimates for eigenvalues of elliptic operators. Arch. Rat. Mech. Anal., **28**, 165–183 (1968).
[2] ——: Lectures on Elliptic Boundary Value Problems. Van Nostrand Mathematical Studies. Princeton (1965).
[3] R. Beals: Asymptotic behavior of the Green's function and spectral function of an elliptic operator. J. Func. Anal., **5**, 485–503 (1970).