55. A Note on Approximate Dimension

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Mityagin has characterized nuclear spaces by the approximate dimension. In an F-space E, namely, E is nuclear iff the approximate dimension of E is zero. (It is known that the approximate dimension is zero if it is finite.) In this note, we shall characterize a Schwarz space by means of metrical dimensions of the same kind. For this purpose, we shall define more general approximate dimensions in an F-space E. An F-space E is called a Schwarz space if for every continuous semi-norm $p(x)$, there exists a continuous semi-norm $q(x)$ such that $U_q=\{x \in E, q(x) \leqq 1\}$ is totally bounded by the semi-norm $p(x)$. For subsets S and K of E, we shall define $N(K, \varepsilon S)$ as usual:

$$N(K, \varepsilon S)=\inf\left\{N: \bigcup_{k=1}^{N}(x_k+\varepsilon S) \supset K, x_k \in E\,; k=1,2,\cdots,N\right\}$$

for a real number $\varepsilon>0$.

An F-space E is a Schwarz space iff for every continuous semi-norm $p(x)$, there exists $q(x)$ such that $N(U_q, \varepsilon U_p)<+\infty$ for all $\varepsilon>0$.

Now, we shall consider two finite valued non-decreasing functions Φ, Ψ, each of which is defined on a sufficient large part of real numbers (i.e. $[\alpha, \infty)$ for some α), such that $\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\Psi(t)=+\infty$. Let $\{U_n\}_{n=1,2,\ldots}$ be any fundamental system of convex neighborhoods of zero in an F-space E. We shall define now another approximate dimension of E by Φ and Ψ as follows:

$$\rho_{\Phi,\Psi}(E)=\sup_k \inf_m \overline{\lim_{\varepsilon\to 0}} \frac{\Phi(N(U_m, \varepsilon U_k))}{\Psi(1/\varepsilon)}.$$

Since $\bigcap_{n=1}^{\infty} U_n=\{0\}$, it is easy to see that $\rho_{\Phi,\Psi}$ is determined uniquely by the topology of E (i.e. independent of the choice of $\{U_n\}_{n=1,2,\ldots}$).

Theorem. *An F-space E is a Schwarz space iff there exist non-decreasing finite valued functions Φ and Ψ with $\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\Psi(t)=+\infty$ such that $\rho_{\Phi,\Psi}(E)<+\infty$.*

Proof. It is easy to see that if $\rho_{\Phi,\Psi}(E)<+\infty$, then E is a Schwarz space. Suppose that E is a Schwarz space. Let $\{U_n\}_{n=1,2,\ldots}$ be a fundamental system of nbd. of zero in E which consists of convex sets. By assumption, we can find $k_n>n$ such that $N(U_{k_n}, \varepsilon U_n)<\infty$ for all $\varepsilon>0$. Let us define

$$f_n(1/\varepsilon)=N(U_{k_n}, \varepsilon U_n) \qquad \text{for} \quad 0<1/\varepsilon<\infty.$$

$f_n(1/\varepsilon)$ is a non-decreasing non-negative function with respect to $1/\varepsilon$ and greater than 1. Let m be a positive integer. For $\varepsilon>0$ with $m-1$

$<1/\varepsilon \leqq m$ $(m=1, 2, \cdots)$, we shall define

$$\Psi(1/\varepsilon)=m \sup_{1\leqq j\leqq m} f_j(1/\varepsilon).$$

Then, $\Psi(1/\varepsilon)$ is defined on $(0, \infty)$ and non-negative non-decreasing function with $\lim_{\varepsilon\to 0} \Psi(1/\varepsilon)=+\infty$. By definition, we have

$$\frac{f_n(1/\varepsilon)}{\Psi(1/\varepsilon)} \leqq \frac{1}{m} \qquad \text{for } n\leqq m \text{ and } m-1<1/\varepsilon.$$

Hence, we have

$$\overline{\lim_{\varepsilon\to 0}} \frac{f_n(1/\varepsilon)}{\Psi(1/\varepsilon)}=0.$$

As a consequence, defining $\Phi(t)=t$,

$$\overline{\lim_{\varepsilon\to 0}} \frac{\Phi(N(U_{k_n}, \varepsilon U_n))}{\Psi(1/\varepsilon)}=0 \qquad \text{for all } n=1, 2, \cdots,$$

that is $\rho_{\Phi,\Psi}(E)=0$.

Remark. In the above theorem, we can suppose $\Psi(t)=\log t$. It may be also supposed that Φ and Ψ are bounded intervals instead of "non-decreasing".

Let $(a_{n,p})$ be an infinite matrix $0<a_{n,p}<\infty$, $a_{n,p}\leqq a_{n,p+1}$ $(p=1, 2\cdots)$. The sequence space $L(a_{n,p})=\{\xi=(\xi_n) : |\xi|_p=\sum_{n=1}^{\infty}|\xi_n|\, a_{n,p}<\infty \text{ for all } p\}$ is called a Köthe space. $L(a_{n,p})$ is an F-space by countable semi-norms $|\xi|_p$ $(p=1, 2, \cdots)$.

Proposition 1. *For every non-descreasing function Φ, Ψ, with $\lim_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\Psi(t)=+\infty$, there exists a Köthe space $L(a_{n,p})$ which is a Schwarz space at the same time such that $\rho_{\Phi,\Psi}(L(a_{n,p}))=+\infty$.*

Proof. It is easy to see that for every Ψ, there exists Ψ_1 such that $\Psi_1(t)\geqq\Psi(t)$ and

(*) $\Psi_1(t)/\Psi_1(s)\geqq t/s$ if t, s and t/s are sufficiently large.

Let us define Ψ^* for Ψ_1 such that

$$\Psi^*(s)=t_s-1 \qquad \text{for } t_s>1$$
$$\Psi^*(s)=0 \qquad \text{for } 0\leqq t_s\leqq 1$$

where $t_s=\inf\{t>0 ; \Psi_1(t)\geqq s\}$ for s with $0<s<\infty$. Then, $\Psi^*(s)$ is non-decreasing and $\lim_{s\to\infty}\Psi^*(s)=\infty$ with $\Psi_1\circ\Psi^*(s)\leqq s$ for sufficiently large s, where $\Psi_1\circ\Psi^*(s)=\Psi_1(\Psi^*(s))$.

Since

$$\frac{\Phi(N(V, \varepsilon U))}{\Psi(1/\varepsilon)} \geqq \frac{\Phi(N(V, \varepsilon U))}{\Psi_1(1/\varepsilon)} \geqq \frac{\Psi_1\circ\Psi^*\circ\Phi(N(V, \varepsilon U))}{\Psi_1(1/\varepsilon)}$$
$$\geqq \frac{\Psi^*\circ\Phi(N(V, \varepsilon U))}{1/\varepsilon} \geqq \frac{\log\circ\Psi^*\circ\Phi(N(V, \varepsilon U))}{\log(1/\varepsilon)}$$

for sufficient small $\varepsilon>0$, we need to prove the proposition only in the case where Φ is arbitrary and $\Psi(1/\varepsilon)=\log(1/\varepsilon)$.

Let $a_{n,p}=[\Phi(n)]^{p-1}$ for $\Phi(n)>1$ and $a_{n,p}=2^{p-1}$ for $\Phi(n)\leqq 1$ and $U_p=\{\xi=(\xi_n) ; |\xi|_p=\sum_{n=1}^{\infty}|\xi_n|\, a_{n,p}\leqq 1\}$. Since $\lim_{n\to\infty} a_{n,p}/a_{n,q}=0$ for $q>p$, the Köthe space $L(a_{n,p})$ is a Schwarz space. By easy calculations, we have $N(U_q, \varepsilon U_p) \geqq e^{n(e\varepsilon)}$, where $q>p$ and $n(e\varepsilon)=\sup\{n; a_{n,p}/a_{n,q} \geqq e\varepsilon\}$. Since we can take

$$\varepsilon=\frac{1}{e(\Phi(n))^{q-p}} \qquad \text{for } n=n(e\varepsilon)$$

such that $\varepsilon \to 0$, we have

$$\overline{\lim_{\varepsilon\to 0}} \frac{\Phi(N(U_q, \varepsilon U_p))}{\log(1/\varepsilon)} \geqq \overline{\lim_{\varepsilon\to 0}} \frac{\Phi\circ\log N(U_q, \varepsilon U_p)}{\log(1/\varepsilon)} \geqq \overline{\lim_{\varepsilon\to 0}} \frac{\Phi(n)}{\log(1/\varepsilon)}$$
$$=\overline{\lim_{\varepsilon\to 0}} \frac{(1/e)^{1/q-p}(1/\varepsilon)^{1/q-p}}{\log(1/\varepsilon)}=+\infty.$$

Hence, $\rho_{\Phi,\Psi}(E)=+\infty$ for $\Psi(t)=\log(t)$.

Corollary 1. *Let* $\Phi_1(t) \geqq \Phi_2(t) \geqq \cdots$ *and* $\Psi_1(t) \leqq \psi_2(t) \leqq \cdots$, *where* Φ_n, Ψ_n *are non-decreasing functions with* $\lim_{t\to\infty} \Phi_n(t)=\lim_{t\to\infty} \Psi_n(t)=+\infty$. *Then, there exists a Köthe space* $E=L(a_{n,p})$ *which is a Schwarz space at the same time such that*

$$\rho_{\Phi_n,\Psi_m}(E)=+\infty \qquad \text{for all } n, m=1,2,\cdots.$$

Corollary 2. *Let*

$$\Phi_n(t)=\overbrace{\log\circ\log\circ\cdots\circ\log}^{n}(t) \quad \textit{and} \quad \Psi_m(t)=\overbrace{\exp\circ\cdots\circ\exp}^{m}(t).$$

Then there exists a Köthe space $E=L(a_{n,p})$ *which is a Schwarz space at the same time such that*

$$\rho_{\Phi_n,\Psi_m}(E)=+\infty \qquad \textit{for all } n, m=1,2,\cdots.$$

Proposition 2. *Let K be a compact set in an F-space E. Then*

$$\rho_{\Phi,\Psi}(E) \leqq \tilde{\rho}_{\Phi,\Psi}(E)=\sup_{K,U} \overline{\lim_{\varepsilon\to 0}} \frac{\Phi(N(K, \varepsilon U))}{\Psi(1/\varepsilon)}$$

If, moreover $\lim_{t\to\infty} \Psi(\rho t)/\Psi(t)=1$ *for* $\rho \geqq 1$, *then* $\rho_{\Phi,\Psi}(E)=\tilde{\rho}_{\Phi,\Psi}(E)$.

Proof. Suppose that $\rho_{\Phi,\Psi}(E)>\tilde{\rho}_{\Phi,\Psi}(E)$. We must have $\tilde{\rho}_{\Phi,\Psi}(E)=M<\infty$. Let $\{U_p\}_{p=1,2,\dots}$ be a basis with $U_p \supset U_{p+1}$ and $\bigcap_{p=1}^\infty U_p=\{0\}$. By assumption, we can find a nbd. of zero U, finite sets $K_p \subset U_p$ and $\varepsilon_p \downarrow 0$ for $p=1,2,\cdots$ such that

$$\frac{\Phi(N(K_p, \varepsilon_p U))}{\Psi(1/\varepsilon_p)} \geqq M+\delta \qquad \text{for some } \delta>0.$$

But, for a compact set $K=\bigcup_{p=1}^\infty K_p \cup \{0\}$, we have

$$\overline{\lim_{\varepsilon\to 0}} \frac{\Phi(N(K, \varepsilon U))}{\Psi(1/\varepsilon)} \geqq M+\delta.$$

This is a contradiction, since $\tilde{\rho}_{\Phi,\Psi}(E)=M$.

Suppose that $\rho_{\Phi,\Psi}(E)=C<+\infty$. Then, for every $\delta>0$ and for all U_k, there exists U_m such that

$$\overline{\lim_{\varepsilon\to 0}} \frac{\Phi(N(U_m, \varepsilon U_k))}{\Psi(1/\varepsilon)} < C+\delta.$$

Let K be an arbitrary compact set in E. We can find $\rho \geqq 1$ such that $\rho U_m \supset K$. If $\lim_{t\to\infty} \Psi(\rho t)/\Psi(t)=1$ for $\rho \geqq 1$,

$$\varlimsup_{\varepsilon\to 0}\frac{\Phi(N(K,\varepsilon U_k))}{\Psi(1/\varepsilon)}\leqq\varlimsup_{\varepsilon\to 0}\frac{\Phi(N(\rho U_m,\varepsilon U_k))}{\Psi(1/\varepsilon)}=\varlimsup_{\varepsilon\to 0}\frac{\Phi(N(U_m,(\varepsilon/\rho)U_k)}{\Psi(1/\varepsilon)}$$
$$=\varlimsup_{\varepsilon\to 0}\left\{\frac{\Phi(N(U_m,(\varepsilon/\rho)U_k))}{\Psi(\rho/\varepsilon)}\right\}\left\{\frac{\Psi(\rho/\varepsilon)}{\Psi(1/\varepsilon)}\right\}<C+\delta.$$

Hence, we have $\tilde{\rho}_{\Phi,\Psi}(E)\leqq\rho_{\Phi,\Psi}(E)$. q.e.d.

References

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