87. *On H-closedness and the Wallman H-closed Extensions. II*[*)]

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1. Introduction. By an extension of a space X is meant a space containing a dense set homeomorphic to X (also denoted by X). A point in the extension not belonging to X is represented by a family of closed sets in X with PFIP which consists of the intersections of X and the closures of the neighborhoods of the point. The collection of all maximal families of closed sets in X with PFIP and suitable topology then constitutes an H-closed extensions $\omega(X)$ of X, called the Wallman H-closed extensions and possesing properties similar to those of the Stone-Čech compactification $\beta(T)$ of a completely regular space T. In particular, continuous functions on X can be continuously extended over $\omega(X)$ and there is a variant of the Stone-Čech theorem [8, p. 153] for Hausdorff spaces.

There are two kinds of normal bases for spaces in literature: one is given by Fan and Gottesman for compactificating regular spaces [4] and the other is employed by Frink to identify complete regularity [6]. These bases are, in fact, equivalent in regular spaces. A new concept, called pseudo-normality which is similar to but more general than normality, is introduced as a characterization of complete regularity. The Fan-Gottesman compactification X^* of a completely regular space X is homeomorphic to the Stone-Čech compactification βX and is also homeomorphic to Aleksandrov $\alpha' X$ [1, p. 405].

The Stone-Weierstrass approximation theorem and the Tietze extension theorem will be generalized to Hausdorff spaces. Aleksandrov [2, Surveys, p. 54] and Pomonarov raised the question: for each completely regular space T whether the Stone-Weierstrass theorem holds in the Wallman H-closed extension $\omega(T)$ (topologically equivalent to $\tau(T)$ in [2]). A theorem due to Fan and Gottesman [4] sheds some light on the problem and an affirmative answer is given in § 4.

2. The Wallman *H*-cosed extensions.

Let X be a space, $\mathfrak{C}$ the family of all closed subsets of X, and $W(X)$ the collection of all subfamilies of $\mathfrak{C}$ which possess the PFIP and are

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maximal in $\mathfrak{C}$ relative to this property. Two elements w_1, w_2 of $W(X)$ are said to be equivalent if both of them contain the closures of the neighborhoods of the same point x in X. An equivalence class in $W(X)$ corresponding to a point x is called a fixed end and denoted by $\mathfrak{A}(x)$; an element in $W(X)$ which dose not belong to any fixed end is called a free end and denoted by $\mathfrak{A}$. We denote by $\omega(X)$ the collection of all fixed and free ends in X. For an open subset U of X let $U^*=\{\mathfrak{A}(x);\ x\in U\}$. We introduce the following topology for $\omega(X)$, called Katětov topology: the neighborhoods for fixed ends $\mathfrak{A}(x)$ are U^* if $x\in U$ and for free ends $\mathfrak{A}$ are U^*, where U is the interior of a closed set A belonging to $\mathfrak{A}$. The space $\omega(X)$ with Katětov topology is H-closed and the subspace consisting of all $\mathfrak{A}(x)$ is homeomorphic to X (also denoted by X). Moreover, the H-closed $\omega(X)$ has the following properties: X is dense in $\omega(X)$, X is open in $\omega(X)$, and $\omega(X)-X$ is discrete [7, p. 45].

Lemma 1. *Every bounded real-valued continuous function f on X ban be continuously extended over $\omega(X)$.*

Proof. Suppose that f cannot be continuously extended at $\mathfrak{A}\in\omega(X)$. Then there is an $\varepsilon>0$ such that to the interior U_α each member A_α of $\mathfrak{A}$ there are $b_\alpha=\sup_{x\in U_\alpha} f(x)$ and $a_\alpha=\mathrm{imf}_{x\in U_\alpha} f(x)$ satisfying the condition $b_\alpha-a_\alpha>\varepsilon$. It is clear that for two members A_α, A_β of $\mathfrak{A}$, $b_\beta-a_\alpha>\varepsilon$, since $A_\alpha\cap A_\beta=A_{\alpha\beta}\in\mathfrak{A}$ and $b_\beta-a_\alpha\geq b_{\alpha\beta}>\varepsilon$. Let L be the least upper bound of $\{a_\alpha\}$ and M the greatest lower bound of $\{b_\alpha\}$. Then $M-L\geq\varepsilon$. $P=\{x;f(x)\geq M-\varepsilon/3\}$ and $Q=\{x;f(x)\leq L+\varepsilon/3\}$ both intersect each member of $\mathfrak{A}$ in sets containing non-vacuous open sets and therefore belong to $\mathfrak{A}$. But $P\cap Q=\phi$, contradicting to the definition of $\mathfrak{A}$.

Corollary. *Every unbounded real-valued continuous function on X can be continuously extended to an extended continuous function over* [see [11] for proof].

If $C(\omega(X))$ is the algebra of all bounded real-valued continuous function on $\omega(X)$, then $\omega(X)$ can be decomposed into disjoint closed subset $S(x)=\{x;f(x)=f(x_0)$ for all $f\in C(\omega(X)), x, x_0\in\omega(X)\}$. A set of $S(x)$ is defined to be open if the union of the $S(x)$'s in the set is open in $\omega(X)$. Then the mapping $\rho: x\to S(x)$ for $x\in\omega(X)$ is continuous and $\{S(x);x\in\omega(X)\}$ from an H-closed space $\Omega(X)$.

The following is a variant of the Stone-Čech theorem [8, p. 153].

Theorem 1. *If X is a space separated by $C(X)$, the algebra of all bounded real-valued continuous functions on X, and f is acontinuous function on X, to an H-closed space Y, separated by $C(Y)$, then there is a pseudo-continuous extension of f over $\Omega(X)$.*

Proof. Let $F(X)$ be the family of all continuous functions on X to the closed unit interval Q and $Q^{F(X)}$ the product of the unit interval Q taken $F(X)$ times. Then $Q^{F(X)}$ is compact and the evaluation map carries an element x of X into the element $l(x)$ of $Q^{F(X)}$ whose f-th coordinate is $f(x)$ for each f in $F(X)$. By Theorem 5 in Part I and Lemma 1, $\Omega(X)$ is pseudo-homeomorphic to $K(X)$ which is a closed subset of $Q^{F(X)}$, and Y is pseudo-homeomorphic to $K(Y)\subset Q^{F(Y)}$. A function f^* on $F(Y)$ to $F(X)$ is induced by the given f if we define $f^*(a)=a\circ f$ for each a in $F(Y)$. Define f^{**} on $Q^{F(X)}$ to $Q^{F(Y)}$ by letting $f^{**}(q)=q\circ f^*$ for each $q\in Q^{F(X)}$. Let i be the embedding map of X into $\Omega(X)$ and let h and g the evalution maps of $\Omega(X)$ and Y into $K(\Omega(X))$ and $K(Y)$ respectively Then $g^{-1}\circ f^{**}$ is the required pseudo-continuous extension of $f\circ h^{-1}\circ l^{-1}$.

3. Complete regularity.

Fan and Gottesman [4] showed that a regular space with a normal base is completely regular; Frink's condition for complete regularity of a T_1 space is to posses a normal base for closed sets [6]. It will be shown here that two concepts of normal base are equivalent for regular spaces, that is, the existence of one kind of normal base implies the existence of the other. We call a regular space paranormal if it has Fan-Gottesman base and seminormal if there exists Frink normal base for closed sets. A concept which is similar to but mor general than normality, called pseudo-normality, is introduced to characterize complete regularity. In fact, the four conditions (1) Complete regularity, (2) Paranormality, (3) Seminormality and (4) Pseudo-normality are equivalent for regular spaces.

Definition. A space is called *pseudo-normal* if for any two open sets Q_1, Q_2 with disjoint closures there are disjoint open U_1, U_2 such that $U_1\supset\bar{Q}_1$, $U_2\supset\bar{Q}_2$.

Lemma 2. *If Q_1, Q_2 are two open subsets of a completely regular space T with disjoint closures $\bar{Q}_1, \bar{Q}_2$, there is $f\in C(T)$ which takes the value O on $\bar{Q}_1$ and I on $\bar{Q}_2$.*

Proof. Following Aleksandrov's notation [1], let $\alpha'T$ be the Hausdorff extension of T consisting of all completery regular ends. Then $\alpha'T$ is compact [1, p. 411] and Q_1, Q_2 have disjoint closures $\tilde{Q}_1, \tilde{Q}_2$ in $\alpha'T$. There is $\bar{f}\in C(\alpha'T)$ assuming the value 0 on $\tilde{Q}_1$ and 1 on $\tilde{Q}_2$. The restriction on $\bar{f}$ on T is the required function.

Theorem 2. *Let X be a regular space. The following statements are equivalent:*

(1) *X is completely regular.*

(2) *X is pseudo-normal.*

(3) *X is paranormal.*

Proof. (1) implies (2). It follows from Lemma 2:

(2) implies (3).

Let U, V be two open sets in X such that $\bar{U}\subset V$. Then U and $X-\bar{V}$ are two open sets with disjoint closures and there is a continuous function f with $f(\bar{U})=0$ and $f(x-\bar{U})=1$. Let $W=\{x \in X; f(x)<1\}$. It is clear that $\bar{U} \subset W \subset \bar{W} \subset V$. The open sets in X therefore from a normal base in Fan-Gottesman sense.

(3) implies (1). It is a theorem due to Fan and Gottesman [4].

The following is a consequence of Theorem 2 and Frink's characterization of complete regularity [6, p. 603, Theorem 1].

Corollary 1. *Paranormality and seminormality are equivalent for regular spaces.*

Corollary 2. *The Fan-Gottesman compactification X^* of a completely regular space X with all open sets in X as a normal base coincides with the Stone-Cech compactification βX.*

The proof of Lemma 1 can be used to show that each continuous function on X can be extended over X^*, and the corollary is proved.

4. On the Stone-Weierstrass approximation theorem.

In order to generalize the Stone-Weierstrass approximation theorem to more general spaces, we adopt the definition: The theorem holds in a apace X is the following conditions are satisfied.

(1) The ring $C(X)$ of all continuous functions (bounded or unbounded) on X separates any two points of an extension $E(X)$ of the space X, on which the functions in $C(X)$ can be continuously extended.

(2) For any subring $K(X)$ of $C(X)$ containing constant functions and separating points of $E(X)$ (a function f in $C(X)$ is said "separating two points in $E(X)$" if its extended function $\tilde{f}$ over $E(X)$ sepates the points), each continuous function f on X is the uniform limit of functions in $K(X)$ on each subset of X on which f is bounded.

It is known that the Stone-Weierstrass theorem holds for completely regular spaces with Stone-Čech compactifications as their extensions [12]. We will show that the theorem is also true for Hausdorff spaces with Wallman H-closed extensions.

Banaschewski [3] showed that each completely regular space X has a non-compact extension, a subset of $\omega(X)$, in which the Stone-Weierstrass theorem holds in the sense in [2]. A question was raised by Aleksandrov and Ponomarev: whether the Stone-Weierstrass theorem holds in $\omega(X)$ [2]. An affirmative answer isg iven in Theorem 5.

Theorem 3. *Let $R(X)$ be an algebra of real-valued continuous functions on an H-closed space X containing constant functions and separating the points. Then every continuous function f on X is the limit of a uniformly convergent sequence of functions belonging to $R(X)$.*

This theorem is an anologue of the one given by Stone [9] for compact spaces, and the proof can be carried out almost without change.

Lemma 3. *Let $\mathfrak{U}$ be an open cover of an H-closed space X sepa-*

rated by $C(X)$. Then there exist a finite pseudo subcover of $U_1, \cdots, U_n$ of $\mathfrak{U}$ and n nonnegative real-valued continuous functions $f_1, \cdots, f_n$ on X such that (1) *f_1 vanish outside of $\bar{U}_i$ for $i=1, \cdots, n$, and* (2) *$f_1(x) + \cdots + f_n(x) = 1$ for each $x \in X$.*

Theorem 4. *Let X be a space separated by $C(X)$ and $\Omega(X)$ the Wallman H-closed extension of X as before. If $S_0(X)$ is a self-adjoint subalgebra of the algebra $K(X)$ of all continuous complex-valued functions on X and is contained in a closed subalgebra $S(X)$ of $K(X)$, then $f \in K(X)$ and $\tilde{f} \in \tilde{S}$ on every set of constancy for S_0 on $\Omega(X)$ imply that f belongs to $S(X)$.*

For notations and the proof of the theorem see [12, p. 931] ("pseudo partition of unity" in Lemma 3 is used in lieu of "partition of unity").

Lemma 4. *For each completely regular space X the continuous functions on $\omega(X)$ separate the points.*

Proof. It follows from Lemma 2 that a completely regular space X has a normal base in the Fan and Gottesman sense [4, p. 504] and thus, can be embedded in a compact space X^*. The free ends in $\omega(X)$ are the maxmal binding families in X [see 4 for definition]. By Lemma 1 each bounded continuous function on X can be continuously extended over X^* is also continuous on $\omega(X)$.

Theorem 5. *For a completely regular space $X, \omega(X) = \Omega(X)$ and the Stone-Weierstrass theorem holds in $\omega(X)$.*

The first part of the theorem follows from Lemma 4 and the second part from Theorem 3.

5. Generalization of the Tietze extension theorem.

Lemma 5. *If A and B are two disjoint H-closed subsets of a space X and $C(X)$ separates the points in $A \cup B$, then there is a continuous function f on X such that$f(A)=0$ and $f(B)=1$.*

Theorem 6. *If A is an H-closed subset of a space X and $C(X)$ separates the points in A, then each bounded continuous function f on A to $[-1, 1]$ can be continuously extended to f over X to $[-1, 1]$.*

Proof. Let $C = \{x : f(x) \leq -1/3, x \in A\}$ and $D = \{x : f)x) \geq 1/3, x \in A\}$. Then C and D are disjoint H-closed sets and by Lemma 5 there is f_1 on X to $[-1/3, 1/3]$ such that $f_1(x)$ is $1/3$ on C and $-1/3$ on D. $|f(x) - f_1(x)| \leq 2/3$ for all x in A.

Remark. The condition that $C(X)$ separates the points in Theorems 3, 4, 6 is assumed for simplicity and more general results with slight modifications still held without such restriction.

6. Terminology.

The characterization of pseudo-compactness as the existence of a cluster point for each sequence of open sets was announced about the same time by (1) K. Iseki and S. Kasahara, Proc. Japan Acad., 33

(1957), (2) S. Mardesic and Z. P. Papic., Glasnik Mat.-Fiz. i Astr., 10 (1955), (3) J. D. McKnight, R. W. Bagley and E. H. Connell, Bull. Amer. Math. Soc., 63(1) (1957), apparently under the influence of Hewitt's paper (Trans. Amer. Math. Soc., 64 (1948)). The existence of a cluster point for each sequence of open sets and of a pseudo finite subcover for each countable open cover reveals the similar between pseudo-compactness and countable compactness. On the other hand, the cluster point theorem for each net of open sets, the existence of a pseudo finite subcover for each open cover, and other properties of H-closed spaces (see Theorems 1, 3, 4 in Part I) are just the anologues of the basic theorems for compact spaces. Even though pseudo-compactness has become a standard term, we feel strongly that the appropriate name for pseudo-compactness is "pseudo countable compactness" while H-closed spaces should be called "pseudo compact".

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