79. *On a Convergence Theorem for Sequences of Holomorphic Functions*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

Let D be the unit disk $\{|z|<1\}$ and C be its circumference $\{|z|=1\}$. For two numbers $\alpha, \beta, 0\leqq\alpha<\beta\leqq 2\pi$, we put

$$S(\alpha,\beta)=\text{the sector } \{z=re^{i\theta}\,;\ \alpha\leqq\theta\leqq\beta, 0\leqq r<1\},$$
$$C(\alpha,\beta)=\text{the arc } \{z=e^{i\theta}\,;\ \alpha\leqq\theta\leqq\beta\},$$
$$S_R(\alpha,\beta)=S(\alpha,\beta)\cap\{|z|<R\}, 0<R<1,$$
$$C_R(\alpha,\beta)=\text{the arc } \{z=Re^{i\theta}\,;\ \alpha\leqq\theta\leqq\beta\}.$$

We say that a function $f(z)$, holomorphic *on* $S(\alpha,\beta)$, *belongs to a class* $N_{(\alpha,\beta)}$ if

$$m(r,f\,;\alpha,\beta)=\int_\alpha^\beta \log^+|f(re^{i\theta})|d\theta \text{ is bounded for } 0\leqq r<1.$$

The class $N_{(0,2\pi)}$ is denoted simply by N and called the class of functions *of bounded characteristic* [1].

A function $f(z)$, holomorphic *in* $S(\alpha,\beta)$, is said to *belong to a class* $N^*_{(\alpha,\beta)}$ if $f(z)\in N_{(\alpha+\delta,\beta-\delta)}$ for every $\delta, 0<\delta<(\alpha+\beta)/2$.

It is proved in [2], as a localization of the Fatou's theorem, that

A function $f(z)$, holomorphic in $S(\alpha,\beta)$, can be written as a quotient of two bounded functions in $S(\alpha+\delta,\beta-\delta)$ for every $\delta, 0<\delta<(\alpha+\beta)/2$, if and only if $f(z)$ belongs to $N^*_{(\alpha,\beta)}$. In particular, a function $f(z)$ of the class $N^*_{(\alpha,\beta)}$ has finite angular limits almost everywhere on $C(\alpha,\beta)$, and if $\{z_n\}$ are the zeros of $f(z)$ in $S(\alpha+\delta,\beta-\delta)(\delta>0$ is fixed), we have

$$\Sigma(1-|z_n|)<\infty.$$

In this note we will prove, using the method of [2], a localization of the theorem of Khintchine-Ostrovski [3, p. 83], i.e.,

Theorem 1. *Let a sequence $\{f_n(z)\}\subset N^*_{(\alpha,\beta)}$ satisfy the conditions*: (i)

$$\int_\alpha^\beta \log^+|f_n(re^{i\theta})|d\theta\leqq K, 0\leqq r<1, \tag{1}$$

where K is a constant independent of n and r.

(ii) *There is a set $E\subset C(\alpha,\beta)$, meas$(E)>0$, on which $\{f_n(e^{i\theta})\}$ converges in measure, where $f_n(e^{i\theta})$ denotes the radial limit of $f_n(z)$ at $e^{i\theta}$.*

Then $\{f_n(z)\}$ converges to a function $f(z)$ uniformly on any compact set in $S(\alpha,\beta)$. $f(z)$ is holomorphic in $S(\alpha,\beta)$ and has finite radial limit $f(e^{i\theta})$ at almost every point $e^{i\theta}\in C(\alpha,\beta)$, and $\{f_n(e^{i\theta})\}$ converges in measure to $f(e^{i\theta})$ on the set E.

Proof. We can find a $\delta>0$, such that

$$\text{meas}\,(E\cap C(\alpha+\delta,\beta-\delta))>0.$$

Hence we can suppose the set E is contained in $C(\alpha+\delta,\beta-\delta)$ for a $\delta>0$.

Fix a point z_0, $0<|z_0|<1$, $\arg[z_0]=(\alpha+\beta)/2$.

Suppose we could show that the best harmonic majorant $u_n(z)$ of $\log^+|f_n(z)|$ in $S(\alpha+\delta,\beta-\delta)$ is bounded at the point z_0:

$$u_n(z_0)\leqq K_1, \tag{2}$$

where K_1 is a constant independent of n.

Then, if $z=\mu(\zeta)$ maps $S(\alpha+\delta,\beta-\delta)$ onto the unit disk $|\zeta|<1$, $z_0=\mu(0)$, the function $F_n(\zeta)=f_n(\mu(\zeta))$ satisfies

$$\int_0^{2\pi}\log^+|F_n(\rho e^{i\phi})|\,d\phi=\int_0^{2\pi}\log^+|f_n(\mu(\rho e^{i\phi}))\,d\phi$$
$$\leqq u_n(\mu(0))=u_n(z_0)\leqq K_1,$$

and $\{F_n(e^{i\phi})\}$ converges in measure on the set $E^*=\mu(E)$, meas $(E^*)>0$. In that case, applying the original Khintchine-Ostrovski's theorem to $\{F_n(\zeta)\}$ and comming back to $\{f_n(z)\}$, we will have our result.

It suffices therefore to prove (2).

From

$$\int_\alpha^\beta \log^+|f_n(re^{i\theta})|\,d\theta\leqq K$$

we can find α_n, β_n, $\alpha<\alpha_n\leqq\alpha+\delta$, $\beta-\delta\leqq\beta_n<\beta$, such that

$$\int_0^1\log^+|f_n(re^{i\alpha_n})|\,dr\leqq K/\delta,$$
$$\int_0^1\log^+|f_n(re^{i\beta_n})|\,dr\leqq K/\delta.$$

Let $\omega_n^R(z;e)(z\in S_R(\alpha_n,\beta_n),\ e\subset\partial S_R(\alpha_n,\beta_n))$ be the harmonic measure of the set e at the point z, with respect to $S_R(\alpha_n,\beta_n)$. Let $U_n^R(z)$ be a harmonic function in $S_R(\alpha_n,\beta_n)$ with boundary values

$$U_n^R(t)=\log^+|f_n(t)|,\ t\in\partial S_R(\alpha_n,\beta_n).$$

Then

$$U_n^R(z)=\int_{\partial S_R(\alpha_n,\beta_n)}\log^+|f_n(t)|\,\omega_n^R(z;dt).$$

By Carleman's principle of "Gebietserweiterung" we have [1, p. 74]

$$U_n^R(z_0)\leqq\frac{1}{\pi}\int_{\partial S_R(\alpha_n,\beta_n)}\log^+|f_n(t)|\,d\phi_n(t),$$

where $\phi_n(t)$ is the argument of $(t-z_0)$, measured from $\overline{z_0z_n^*}$, where z_n^* is the foot of the perpendicular to the radius $B_R(\alpha_n)$:

$$B_R(\alpha_n)=\{z=re^{i\alpha_n};\ 0\leqq r\leqq R\}.$$

We write $|t-z_0|=\rho$, $|z_n^*-z_0|=a_n$, $|z_n^*|=b_n$. Then

$$U_n^R(z_0)\leqq\frac{1}{\pi}\left(\int_{B_R(\alpha_n)}+\int_{B_R(\beta_n)}+\int_{C_R(\alpha_n,\beta_n)}\right)=\frac{1}{\pi}(I_1+I_2+I_3).$$

If $t\in B_R(\alpha_n)$ we have, writing $t=re^{i\theta}$,

$$a_n \tan \phi_n + b_n = r.$$

Differentiating,

$$a_n \sec^2 \phi_n d\phi_n = dr,$$

Thus

$$d\phi_n \leqq \frac{1}{a} dr, \tag{3}$$

where $a = |z^* - z_0| \leqq |z_n^* - z_0| = a_n$, in which z^* is the foot of the perpendicular to $B_R(\alpha+\delta)$ from z_0.

Hence

$$\int_{B_R(\alpha_n)} \log^+ |f_n(t)|\, d\phi_n(t) \leqq \int_0^R \log^+ |f_n(re^{i\alpha_n})| \frac{dr}{a}$$

$$\leqq \frac{1}{a}\int_0^1 \log^+ |f_n(re^{i\alpha_n})|\, dr \leqq K/a\delta.$$

Similarly

$$\int_{B_R(\beta_n)} \log^+ |f_n(t)|\, d\phi_n(t) \leqq \frac{K}{a\delta}.$$

If $t \in C_R(\alpha_n, \beta_n)$

$$|dt|^2 = (Rd\theta)^2 = d\rho^2 + (\rho d\phi_n)^2 \geqq \rho^2 d\phi_n^2,$$

hence there is a constant A, independent of R and n, such that

$$d\phi_n \leqq A d\theta. \tag{4}$$

Therefore

$$\int_{C_R(\alpha_n, \beta_n)} \log^+ |f_n(t)|\, d\phi_n(t) \leqq A \int_{\alpha_n}^{\beta_n} \log^+ |f_n(Re^{i\theta})| d\theta \leqq AK.$$

Thus

$$U_n^R(z_0) \leqq \left(\frac{1}{a\delta} + \frac{1}{a\delta} + A\right) K = K_1.$$

As $U_n^R(z)$ increases with R because of subharmonicity of $\log^+ |f_n(z)|$,

$$U_n(z_0) = \lim_{R\to 1} U_n^R(z_0) \leqq K_1.$$

Since

$$u_n(z) \leqq U_n(z) \qquad \text{in } S(\alpha+\delta, \beta-\delta),$$

we have the inequality (2) and our proof is completed. Q.E.D.

Let $m(r) > 0$ be a continuous function of $r, 0 \leqq r < 1$, $\overline{\lim}_{r\to 1} m(r) \leqq \infty$.

The following lemma is easily proved.

Lemma. *Let $\{f_n(z)\}$ be a sequence of functions holomorphic in D, such that*

$$\int_0^{2\pi} \log^+ |f_n(re^{i\theta})|\, d\theta \leqq m(r), 0 \leqq r < 1. \tag{5}$$

Then $\{f_n(z)\}$ forms a normal family in the Montel's sense in D.

Using this lemma, we can prove easily the following version of Theorem 1 for a sequence of functions holomorphic *in D.*

Theorem 2. *Let a sequence $\{f_n(z)\}$, holomorphic in D, satisfy the conditions:*

(i)

$$\int_{\alpha}^{\beta} \log^{+} |f_n(re^{i\theta})| d\theta \leqq K, 0 \leqq r < 1,$$

where K is a constant independent of n and r.

(ii) *there is a set $E \subset C(\alpha, \beta)$*, meas $(E) > 0$, *on which the sequence of radial limits $\{f_n(e^{i\theta})\}$ converges in measure.*

(iii)

$$\int_{0}^{2\pi} \log^{+} |f_n(re^{i\theta})| d\theta \leqq m(r), 0 \leqq r < 1,$$

for a function $m(r)$ as stated above.

Then, $\{f_n(z)\}$ converges to a function $f(z)$ uniformly on any compact set in D. $f(z)$ is holomorphic in D, has radial limit $f(e^{i\theta})$ at almost every point $e^{i\theta} \in C(\alpha, \beta)$, and $\{f_n(e^{i\theta})\}$ converges to $f(e^{i\theta})$ in measure on the set E.

Next we will show: there is a set $E \subset [0, 2\pi]$, meas $(E) > 0$, and a sequence $\{f_n(z)\}$ of holomorphic functions in D, such that

(i)

$$\int_{E} \log^{+} |f_n(re^{i\theta})| d\theta \leqq K, 0 \leqq r < 1.$$

where K is a constant independent of n and r.

(ii) *each $f_n(z)$ has radial limit $f_n(e^{i\theta})$ at almost every point of C, and $\{f_n(e^{i\theta})\}$ converges to 0 on the set E.*

(iii)

$$\int_{0}^{2\pi} \log^{+} |f_n(re^{i\theta})| d\theta \leqq m(r), 0 \leqq r < 1,$$

for a function $m(r)$ as stated in Theorem 2, while $\{f_n(z)\}$ converges at no point in D.

That is, the Khintchine-Ostrowski's theorem can be localized to integrals *over an interval*, but not to integrals over a set of positive measure.

To see this, let $f(z)$ be a holomorphic function in D such that

$$f(z) \neq 0 \text{ in } D,$$

$$\lim_{r \to 1} f(re^{i\theta}) = 0 \text{ for almost every } \theta, 0 \leqq \theta \leqq 2\pi,$$

$$\max |f(z)| < \frac{1}{2\pi} m(r) \text{ for } |z| = r, 0 \leqq r < 1.$$

Such a function is constructed in [4].

For each positive integer N we set

$$E_N = \{\theta ; 0 \leqq \theta \leqq 2\pi, |f(re^{i\theta})| \leqq N \text{ for } 0 \leqq r < 1\},$$

then

$$\text{meas}\left(\bigcup_{N=1}^{\infty} E_N\right) = 2\pi,$$

hence there is an N, meas $(E_N) > 0$.

Put

$$E=E_N,$$

$$f_n(z)=(-1)^n f(z)+\frac{1}{n},\ n=1,2,3,\cdots,$$

then E and $\{f_n(z)\}$ satisfy above conditions (i), (ii), (iii) obviously.

References

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