77. The Theory of Nuclear Spaces Treated by the Method of Ranked Space. II

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In the paper [3], we defined the neighbourhood having a rank in the nuclear space Φ.

Now in this note we shall prove that the space Φ above is a linear ranked space.

§3. **Definition of unit ball.** Following §2, we suppose the mappings $T_{n_i}^{n_{i+1}}$, $i=0,1,2,\cdots$, in the nuclear space Φ. Furthermore, we consider a fixed sequence of real numbers $\{\varepsilon_i\}$ such that

(1) $$\varepsilon_1=1$$

(2) $$2\left(\sum_{k=1}^{\infty}\lambda_{k,n_i,n_{i+1}}\right)\varepsilon_{i+1}\leqq\varepsilon_i$$

(3) $$0<\varepsilon_{i+1}<\varepsilon_i.$$

Then we define $V_i(0,1,m)\equiv U_i(0,\varepsilon_i,m)$ as the unit ball of neighbourhood with rank i in regarding to m.

In particular, we define that the neighbourhood with rank 0, V_0, is always the space Φ.

By the definition of $U_i(0,\varepsilon_i,m)$ in §2, it is easily verified to be $rV_i(0,1,m)=V_i(0,r,m)$ for any $r>0$.

Then we shall call number r the radius of neighbourhood $V_i(0,r,m)$.

Lemma 5. *We have* $V_j(0,1,m)\supseteq V_i(0,1,m)$ *if* $j\leqq i$.

Proof. By Lemma 1, it is clear.

Lemma 6. *We have* $V_i(0,1,m')\supseteq V_i(0,1,m)$ *if* $m'\leqq m$.

Lemma 7. *We have* $V_i(0,r,m)\supseteq V_i(0,r'm)$ *if* $r'\leqq r$.

Now, we shall define the fundamental sequence of neighbourhoods.

Definition 1. When a sequence of neighbourhoods $\{V_{\gamma_i}(0,r_i,m_i)\}$ satisfies the following conditions, it is called the fundamental sequence.

(1) there exists some integer i_0 such that $V_{\gamma_i}(0,r_i,m_i)=V_0$ for $0\leqq i\leqq i_0$,

(2) $\gamma_i\leqq\gamma_{i+1}$ for $i>i_0$ and $\gamma_i\to\infty$,

(3) $r_i\geqq r_{i+1}$ for $i>i_0$ and $r_i\to 0$,

(4) $m_i\leqq m_{i+1}$ for $i>i_0$ and $m_i\to\infty$.

Lemma 8. *If* $\{V_{\gamma_i}(0,r_i,m_i)\}$ *is a fundamental sequence of neighbourhoods, then* $g\in V_{\gamma_i}(0,r_i,m_i)$ *for every* i *implies* $g=0$.

Proof. By Lemma 2, it is clear.

Lemma 9. (1) $V_i(0,r,m)$ *is circled.*

(2) $V_i(0, r, m) + V_i(0, r', m') \subseteqq V_i(0, r+r', \min(m, m'))$.

Proof. (1) Since we have $V_i(0, r, m) = rV_i(0, 1, m) = rU_i(0, \varepsilon_i, m)$ by the definition, it is clear.

(2) The relations $g \in V_i(0, r, m)$ and $g' \in V_i(0, r', m')$ imply

$$\left\| \sum_{k=1}^{\min(m,m')} \lambda_{k,n_{i-1},n_i}(g+g', \varphi_{k,n_i})_{n_i} \varphi_{k,n_{i-1}} \right\|_{n_{i-1}}$$
$$\leqq \left\| \sum_{k=1}^{m} \lambda_{k,n_{i-1},n_i}(g, \varphi_{k,n_i})_{n_i} \varphi_{k,n_{i-1}} \right\|_{n_{i-1}}$$
$$+ \left\| \sum_{k=1}^{m'} \lambda_{k,n_{i-1},n_i}(g', \varphi_{k,n_i})_{n_i} \varphi_{k,n_{i-1}} \right\|_{n_{i-1}} < r\varepsilon_i + r'\varepsilon_i = (r+r')\varepsilon_i.$$

And hence $g+g'$ is contained in $V_i(0, r+r', \min(m, m'))$.

A neighbourhood of the origin in the space Φ is representable, except V_0, in the form $V_i(0, r, m) = rV_i(0, 1, m)$, where $V_i(0, 1, m)$ is the unit ball and number r is a strictly positive real number.

Thus we shall define $V_i(g, r, m) \equiv g + V_i(0, r, m)$ as the neighbourhood of a point g in the space Φ.

Now we shall prove that the space Φ satisfies the three conditions, i.e. (RL_1), (RL_2), and (RL_3) in Washihara [2].

Condition (RL_1). Suppose that $\{V_{\gamma_i}(0, r_i, m_i)\}$ and $\{V_{\gamma'_i}(0, r'_i, m'_i)\}$ are two arbitrary fundamental sequences of neighbourhoods.

By the definition in the paper [3], there exists some integer i_0 such that for all $i > i_0$, $\gamma_i \uparrow \infty$, $r_i \downarrow 0$, $m_i \uparrow \infty$, $\gamma'_i \uparrow \infty$, $r'_i \downarrow 0$ and $m'_i \uparrow \infty$. And then we see that $\min(\gamma_i, \gamma'_i) \uparrow \infty$, $(r_i + r'_i) \downarrow 0$ and $\min(m_i, m'_i) \uparrow \infty$ for all $i > i_0$.

Thus if we set $V_{\gamma''_i}(0, r''_i, m''_i) \equiv V_0$ for $i \leqq i_0$ and

$$V_{\gamma''_i}(0, r''_i, m''_i) \equiv V_{\min(\gamma_i, \gamma'_i)}(0, r_i + r'_i, \min(m, m')) \qquad \text{for } i > i_0,$$

we obtain the fundamental sequence $\{V_{\gamma''_i}(0, r''_i, m''_i)\}$. Since we have

$$V_{\gamma_i}(0, r_i, m_i) + V_{\gamma'_i}(0, r'_i, m'_i) \subseteqq V_{\min(\gamma_i, \gamma'_i)}(0, r_i, m_i) + V_{\min(\gamma_i, \gamma'_i)}(0, r'_i, m'_i)$$
$$\subseteqq V_{\min(\gamma_i, \gamma'_i)}(0, r_i + r'_i, \min(m_i, m'_i)),$$

the proof is complete.

Condition (RL_2) (i). Let $\{V_{\gamma_i}(0, r_i, m_i)\}$ be any fundamental sequence of neighbourhoods. Then $\{\lambda V_{\gamma_i}(0, r_i, m_i)\}$ is also fundamental for any scalar λ, since $\lambda V_{\gamma_i}(0, r_i, m_i) = V_{\gamma_i}(0, |\lambda| r_i, m_i)$.

Condition (RL_2) (ii). For any point g in Φ and $\{\lambda_n\}$ with $\lim \lambda_n = 0$, we shall prove that there exists a fundamental sequence $\{V_{\gamma_i}(0, r_i, m_i)\}$ such that $\lambda_i g \in V_{\gamma_i}(0, r_i, m_i)$.

First, take $\{m_i\}$ to be a sequence of the strictly positive integers with $m_i \uparrow \infty$.

Second, let $\{r'_n\}$ be the sequence of real numbers with $r'_n > 0$ and $r'_n \downarrow 0$, where $r'_1 \equiv |\lambda_1| \, \| \sum_{k=1}^{m_1} \lambda_{k,1,n_1}(g, \varphi_{k,n_1})_{n_1} \varphi_{k,1} \|_1$. Since $\lim \lambda_n = 0$, there exists some integer h_1 such that

$$|\lambda_n| \left\| \sum_{k=1}^{m_2} \lambda_{k,n_1,n_2}(g, \varphi_{k,n_2})_{n_2} \varphi_{k,n_1} \right\|_{n_1} < r'_2 \varepsilon_2, \qquad \text{for all } n > h_1.$$

Similarly, we find some integer h_2 such that $h_2 > h_1$ and

$$|\lambda_n|\left\|\sum_{k=1}^{m_3}\lambda_{k,n_2,n_3}(g,\varphi_{k,n_3})_{n_3}\varphi_{k,n_2}\right\|_{n_2}<r_3'\varepsilon_3, \qquad \text{for all } n>h_2.$$

In the same manner, we obtain the sequence $\{h_n\}$.

Now, we set

$$r_1=2\max_{i=1\cdots h_1}|\lambda_i|\left\|\sum_{k=1}^{m_1}\lambda_{k,1,n_1}(g,\varphi_{k,n_1})_{n_1}\varphi_{k,1}\right\|_1$$

$$r_2=r_2',\ r_3=r_3',\ \cdots.$$

Hence we have

$$V_1(0,r_1,m_1)\ni\lambda_1 g,\lambda_2 g,\cdots,\lambda_{h_1-1}g,$$
$$V_2(0,r_2,m_2)\ni\lambda_{h_1}g,\lambda_{h_1+1}g,\cdots,\lambda_{h_2-1}g,$$
$$V_3(0,r_3,m_3)\ni\lambda_{h_2}g,\lambda_{h_2+1}g,\cdots,\lambda_{h_3-1}g,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

If we set $V_{\gamma_0}(0,r_0,m_0)\equiv V_0$, $V_{\gamma_i}(0,r_i,m_i)\equiv V_1(0,r_1,m_1)$ for $0<i<h_1$ and $V_{\gamma_i}(0,r_i,m_i)\equiv V_2(0,r_2,m_2)$ for $h_1\leqq i<h_2$, etc., we complete the proof.

Condition (RL_3). It is clear from the definition.

Lemma 10. *For any neighbourhood $V_i(0,r,m)$ and any fundamental $\{V_{\gamma_i}(0,r_i,m_i)\}$, there exists some integer j such that*

$$V_i(0,r,m)\supseteq V_{\gamma_j}(0,r_j,m_j).$$

Proof. Since $\gamma_i\uparrow\infty$, $r_i\downarrow 0$ and $m_i\uparrow\infty$, there exists some integer j such that $i<\gamma_j$, $r>r_j$ and $m<m_j$. Hence it is clear.

Definition 2. A sequence $\{g_n\}$ in Φ is called a R-Cauchy sequence of elements in Φ, if there exists a fundamental sequence of neighbourhoods $\{V_{\gamma_i}(0,r_i,m_i)\}$ such that the relations $n\geqq i$ and $m\geqq i$ imply $g_n-g_m\in V_{\gamma_i}(0,r_i,m_i)$.

Definition 3. We say that two R-Cauchy sequence $\{g_n\}$ and $\{f_n\}$ are equivalent, if there exists a fundamental sequence of neighbourhoods $\{V_{\gamma_i}(0,r_i,m_i)\}$ such that $g_i-f_i\in V_{\gamma_i}(0,r_i,m_i)$ for all i. And we write $\{g_n\}\sim\{f_n\}$.

Lemma 11. *The equivalence by Definition 3 is reflexive, symmetric and transitive.*

Proof. We see easily that it is reflexive and symmetric. Then we shall prove that it is transitive. If $\{g_n\}\sim\{g_n'\}$ and $\{g_n'\}\sim\{g_n''\}$, there exist some $\{V_{\gamma_i}(0,r_i,m_i)\}$ and $\{V_{\gamma_i'}(0,r_i',m_i')\}$ such that $g_i-g_i'\in V_{\gamma_i}(0,r_i,m_i)$ and $g_i'-g_i''\in V_{\gamma_i'}(0,r_i',m_i')$ for all i. And then we have

$$g_i-g_i''\in V_{\gamma_i}(0,r_i,m_i)+V_{\gamma_i'}(0,r_i',m_i')\subset V_{\min(\gamma_i,\gamma_i')}(0,r_i+r_i',\min(m_i,m_i')).$$

Since $\{V_{\min(\gamma_i,\gamma_i')}(0,r_i+r_i',\min(m_i,m_i))\}$ is fundamental, this proof is now complete.

Lemma 12. *Suppose $\{g_n\}$ to be a sequence in Φ. Then if to any neighbourhood $V(0,r,m)$ there corresponds some integer N such that the relations $n\geqq N$ and $m\geqq N$ imply $g_n-g_m\in V(0,r,m)$, $\{g_n\}$ is a R-Cauchy sequence.*

Proof. Let $\{V_i(0,r_i,m_i)\}$ be a fundamental sequence of neighbourhoods. Then there exists a sequence of integers $\{N_i\}$ such that the relations $n \geqq N_i$ and $m \geqq N_i$ imply $g_n - g_m \in V_i(0, r_i, m_i)$ and $N_i < N_{i+1}$.

Since $\{V_i(0, r_i, m_i)\}$ is fundamental, there exists an integer j such that $V_i(0, r_i, m_i) = V_0$ for $0 \leqq i < j$ and $V_i(0, r_i, m_i) \neq V_0$ for $i \geqq j$.

Now we set $V_{\gamma_i}(0, r'_i, m'_i) = V_0$ for $0 \leqq i < N_j$, $V_{\gamma_i}(0, r'_i, m'_i) = V_j(0, r_j, m_j)$ for $N_j \leqq i < N_{j+1}$ and $V_{\gamma_i}(0, r'_i, m'_i) = V_{j+1}(0, r_{j+1}, m_{j+1})$ for $N_{j+1} \leqq i < N_{j+2}$, etc.,

Then $\{V_{\gamma_i}(0, r'_i, m'_i)\}$ is fundamental such that the relations $n \geqq i$ and $m \geqq i$ imply $g_n - g_m \in V_{\gamma_i}(0, r'_i, m'_i)$.

Hence $\{g_n\}$ is a R-Cauchy sequence.

Lemma 13. *The sequence $\{g_n\}$ in Φ is a R-Cauchy sequence if and only if, for any $V(0, r, m)$ there exists some integer N such that the relations $n \geqq N$ and $m \geqq N$ imply $g_n - g_m \in V(0, r, m)$.*

Proof. It is clear.

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