170. *On Some Subgroups of the Group Sp(2n, 2)*

By Eiichi BANNAI[*)]
University of Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., Dec. 13, 1971)

Introduction. We say that a subgroup H of a group G is of rank 2, if the number of double cosets $H\backslash G/H$ is equal to 2. Any subgroup of rank 2 of G is the stabilizer of a point of some doubly transitive permutation representation of G, and vice versa. It is known that the symplectic group $Sp(2n,2)$ has two kinds of subgroups of rank 2 of index $2^{n-1}(2^n+1)$ and $2^{n-1}(2^n-1)$ which are isomorphic to the groups $0(2n,2,+1)$ and $0(2n,2,-1)$, respectively. Here $0(2n,2,+1)$ and $0(2n,2,-1)$ denote the orthogonal group of index n and $n-1$ defined over a field with 2 elements, respectively.

The purpose of this note is to give an outline of the proof of the following Theorem 1 which asserts that the two kinds of subgroups mentioned above are the only subgroups of rank 2 of the group $Sp(2n,2)$. Details will be published elsewhere.

Theorem 1. *Let H be a subgroup of rank 2 of the group $Sp(2n,2)$, $n\geqslant 3$. Then either*
1) *H is of index $2^{n-1}(2^n+1)$ and is isomorphic to the group $0(2n,2,+1)$, or*
2) *H is of index $2^{n-1}(2^n-1)$ and is isomorphic to the group $0(2n,2,-1)$.*

§1. The group $Sp(2n,2)$.

We may define $G=Sp(2n,2)$, the symplectic group defined over the finite field $GF(2)$, by

$$G=\left\{X\in GL(2n,2)\,;\,{}^tXJX=J,\ \text{with } J=\begin{pmatrix} & I_n\\ I_n & \end{pmatrix}\right\}.$$

Here I_n denotes the $n\times n$ identity matrix, and the unwritten places of any matrix always represent 0. The group $G=Sp(2n,2)$ is simple if $n\geqslant 3$.

Let us define some subgroups of the group G as follows:

$$Q=\left\{X\in GL(2n,2)\,;\,X=\begin{pmatrix} I_n & B\\ & I_n\end{pmatrix},\ \text{with } {}^tB=B\right\},$$

$$L=\left\{X\in GL(2n,2)\,;\,X=\begin{pmatrix} A & \\ & {}^t\!A^{-1}\end{pmatrix},\ \text{with } A\in GL(n,2)\right\},$$

$$R=\left\{X\in GL(2n,2)\,;\,X=\begin{pmatrix} A & \\ & {}^t\!A^{-1}\end{pmatrix},\ \text{where } A \text{ is any upper triangular unipotent } n\times n \text{ matrix}\right\}.$$

*) Supported in part by the Fujukai Foundation.

Then L and R normalizes Q. We set $B=RQ$ (semi-direct product). Let π denote the canonical projection $B=RQ\to R=B/Q$.

Now, $Z(B)$(=the center of B) consists of 4 elements 1, u_1, u_2 and u_3, where $u_1=I_{2n}+e_{1,n+1}$, $u_2=I_{2n}+e_{2,n+1}+e_{1,n+2}$ and $u_3=u_1u_2=I_{2n}+e_{1,n+1}+e_{1,n+2}+e_{2,n+1}$. Here the e_{ij} denote the matrix whose (i,j)-entry is 1 and other entries are all 0.

We can also regard the group $G=Sp(2n,2)$ as the Chevalley group of type (C_n) defined over the field $GF(2)$. Naturally G has a Tits system (i.e., BN-pair) whose Coxeter diagram (W,R) is as follows:

$$\underset{w_1}{\circ}-\underset{w_2}{\circ}-\underset{w_3}{\circ}-\cdots-\underset{w_{n-1}}{\circ}=\underset{w_n}{\circ},\qquad R=\{w_1,w_2,\cdots,w_n\}.$$

For any subset $J\subset R$, the groups W_J and G_J are defined by

W_J=the group generated by the w_i with $w_i\in J$,

$G_J=\bigcup_{w\in W_J}BwB$, where B denotes the Borel subgroup of the Tits system.

Now, we can show that we may take the subgroup $RQ=B$ for the Borel subgroup, the group $C_G(u_1)$(resp. $C_G(u_2)$, $C_G(u_3)$) for the subgroup $G_{R-\{w_1\}}$(resp. $G_{R-\{w_2\}}$, $G_{R-\{w_1,w_2\}}$) and the group LQ for the subgroup $G_{R-\{w_n\}}$ of a fixed Tits system of G.

§2. Outline of the proof of Theorem 1.

Let H be a subgroup of rank 2 of the group $G=Sp(2n,2)$, and let χ be the irreducible character of G such that $(1_H)^G=1_G+\chi$, where 1_H and 1_G denote the identity characters of the groups H and G respectively and $(1_H)^G$ denotes the induced character of 1_H to G. We fix these notations throughout this note.

To avoid the complication of the statements and to clarify the method of the proof, we always assume that $n\geqslant 7$ in the rest of this note. The proof for $n=3$, 4, 5 and 6 is done in the same way as that for $n\geqslant 7$ in broad outline although some special treatments are needed, and is omitted in this note.

The proof of Theorem 1 is completed using the following chain of Lemmas 1 to 5.

Lemma 1. $|G:H|\leqslant 2^{2n}$, *consequently* $\chi(1)\leqslant 2^{2n}-1$.

Proof of Lemma 1. Since $|G:C_G(u_1)|=2^{2n}-1$, we have the assertion by a lemma of Ed. Maillet (Cf. [1], Lemma 3).

Lemma 2. *H contains an element x which is conjugate to one of the elements u_1, u_2 and u_3.*

To prove Lemma 2, we need Propositions A and B.

Proposition A (This is proved by making use of the results in J. A. Green [3]. Here we use the assumption that $n\geqslant 7$). *The irreducible characters of $GL(n,2)$ whose degrees are $\leqslant 2^{2n-2}$ are as follows:*

1) $I_1[n]$, of degree 1,

2) $I_1[n-1,1]$, of degree $2(2^{n-1}-1)$,

3) $I_1[n-2, 2]$, of degree $\frac{2^2}{3}(2^n-1)(2^{n-3}-1)$,

4) $-I_2[1] \circ I_{n-2}[1]$, of degree $\frac{1}{3}(2^n-1)(2^{n-1}-1)$.

(For the notation, see [3]. Since the 1-simplex and 2-simplex are unique in this case, the subscripts about simpleces are omitted.)

Proposition B (This is proved by Proposition A together with some additional considerations). *Any subgroup K of $GL(n, 2)$ whose index is $\leqslant 2^{2n-2}$ is conjugate to one of the following subgroups:*

1) $GL(n, 2)$,

2) $$G^{(1)}=\left\{X \in GL(n,2)\,;\, X=\left(\begin{array}{c|c} 1 & \\ \hline * & \\ \vdots & A \\ * & \end{array}\right),\ A \in GL(n-1,2)\right\},$$

3) $$G^{(2)}=\left\{X \in GL(n,2)\,;\, X=\left(\begin{array}{c|c} A & \\ \hline ** & \\ \vdots & B \\ ** & \end{array}\right),\ A \in GL(2,2),\ B \in GL(n-2,2)\right\},$$

4) $$G^{(n-2)}=\left\{X \in GL(n,2)\,;\, X=\left(\begin{array}{c|c} A & \\ \hline *\cdots* & B \\ *\cdots* & \end{array}\right),\ A \in GL(n-2,2),\ B \in GL(2,2)\right\},$$

5) $$G^{(n-1)}=\left\{X \in GL(n,2)\,;\, X=\left(\begin{array}{c|c} A & \\ \hline *\cdots* & 1 \end{array}\right),\ A \in GL(n-1,2)\right\}.$$

Proof of Lemma 2. Let us assume that the assertion is false. Clearly we have $|LQ : LQ \cap H| \leqslant 2^{2n}$ by Lemma 1, and we have $|L : \pi(LQ \cap H)| \leqslant 2^{2n-2}$ from the above assumption. Thus we may assume that $\pi(LQ \cap H)$ is one of the subgroups listed in Proposition B. Clearly we have $|Q : Q \cap H| \leqslant 2^{2n}$, and the group $Q \cap H$ must be invariant under the action of $\pi(LQ \cap H)$. However, we can show that for every group $\pi(LQ \cap H)$ listed in Proposition B, any subgroup of Q which is of index $\leqslant 2^{2n}$ and invariant under the action of $\pi(LQ \cap H)$ contains an element which is conjugate in G to one of u_1, u_2 and u_3, a contradiction. Thus Lemma 2 is proved.

Lemma 3. *The irreducible character χ appears in $(1_{G_{R-\{w_1,w_2\}}})^G$.*

Proof of Lemma 3. By Lemma 2 H contains an element x which is conjugate in G to one of u_1, u_2 and u_3. Let us assume that χ does not appear in $(1_{G_{R-\{w_1,w_2\}}})^G$. Then 1_G is the only irreducible character of G which appears both in $(1_{C_G(x)})^G$ and $(1_H)^G$, hence a theorem of D. E. Littlewood and J. S. Frame shows that $G=C_G(x)H$. Hence we have $|G : C_G(x)|=|C_G(x)H : H|=|H : H \cap C_G(x)|=|H : C_H(x)|$. Now, the subgroup generated by the elements which are conjugate in G to x is a subgroup of $H(\lneqq G)$, and moreover this subgroup must be a normal subgroup of G. This is a contradiction, and Lemma 3 is proved.

Lemma 4. *The irreducible character χ is equal to either the irreducible character χ_1 or χ_2, where χ_1 and χ_2 are the non-identity irreducible characters of G appearing in $(1_{G_{R-\{w_1\}}})^G$. Moreover, the index of H in G is either $2^{n-1}(2^n+1)$ or $2^{n-1}(2^n-1)$.*

To prove Lemma 4, we need Propositions C and D.

Proposition C. *$(1_{G_{R-\{w_1\}}})^G$ is decomposed into 3 irreducible characters whose multiplicities are all 1. $(1_{G_{R-\{w_2\}}})^G$ is decomposed into 6 irreducible characters whose multiplicities are all 1. $(1_{G_{R-\{w_1,w_2\}}})^G$ is decomposed into 8 irreducible characters of which 5 are of multiplicities 1 and 3 are of multiplicities 2.*

Proposition C is proved by looking at the characters of the Weyl group. (See [2], there it is proved that there exists a bijection between the set of irreducible characters of W and the set of irreducible characters of G appearing in $(1_B)^G$ which preserves the multiplicities in $(1_{W_J})^W$ and $(1_{G_J})^G$.)

Proposition D. *The degree of 6 irreducible characters of G appearing in $(1_{G_{R-\{w_2\}}})^G$ are as follows:*

1) 1
2) $(2^n-1)(2^{n-1}+1)$
3) $(2^n+1)(2^{n-1}-1)$
4) $\frac{2}{9}(2^n+1)(2^n-1)(2^{n-1}+1)(2^{n-3}-1)$
5) $\frac{2}{9}(2^n+1)(2^n-1)(2^{n-1}-1)(2^{n-3}+1)$
6) $\frac{8}{9}(2^n+1)(2^n-1)(2^{n-2}+1)(2^{n-2}-1)$.

Moreover the first three members are those characters appearing in $(1_{G_{R-\{w_1\}}})^G$ and are respectively 1_G, χ_1 and χ_2.

Proposition D is proved by the method of intersection matrices in D. G. Higman [4]. Note that the intersection matrix of the permutation group $(G, G/G_{R-\{w_2\}})$ is given as follows: rank is 6 and the subdegrees are $l_0=1$, $l_1=6(2^{2n-4}-1)$, $l_2=16/3(2^{2n-4}-1)(2^{2n-6}-1)$, $l_3=2^{4n-5}$, $l_4=6\cdot 2^{2n-4}$ and $l_5=3\cdot 2^{2n-2}(2^{2n-4}-1)$; the intersection matrix $M=(\mu_{ij}^{(1)})$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 6(2^{2n-4}-1) & 2^{2n-4}+1 & 9 & 0 & 2^{2n-4}-1 & 1 \\ 0 & 2^{2n-3}-8 & 3\cdot 2^{2n-5}-15 & 0 & 0 & 2^{2n-5}-2 \\ 0 & 0 & 0 & 3(2^{2n-4}-1) & 0 & 2^{2n-3} \\ 0 & 2^{2n-4} & 0 & 0 & 2^{2n-4}-1 & 2 \\ 0 & 2^{2n-3} & 9\cdot 2^{2n-5} & 3(2^{2n-4}-1) & 2^{2n-4}-4 & 7\cdot 2^{2n-5}-7 \end{pmatrix};$$

the eigen values of M are $\theta_0=6(2^{2n-4}-1)$, $\theta_1=(2^{n-1}-5)(2^{n-2}+1)$, $\theta_2=(2^{n-1}+5)(2^{n-2}-1)$, $\theta_3=-3(2^{n-2}+1)$, $\theta_4=3(2^{n-2}-1)$ and $\theta_5=-3$. We have the degrees by [4], Theorem 5.5. The assertion of the latter part is easily verified.

Proof of Lemma 4. Let ψ_1, ψ_2 be the irreducible characters of G which appear in $(1_{G_{R-\{w_1,w_2\}}})^G$ but not in $(1_{G_{R-\{w_2\}}})^G$. Now, we can show using Propositions C and D that if $\psi_1(1)$ or $\psi_2(1)$ is odd then both $\psi_1(1)$ and $\psi_2(1)$ are $\geqslant 2^{2n}$. Thus Lemma 4 is immediately proved by Propositions C and D together with the fact that $\chi(1)$ is odd. Because, if $\chi(1)$ is even, then H contains a Sylow 2-subgroup, and so H is a parabolic subgroup. However, we can see that there exists no parabolic subgroup of rank 2. This is proved by looking at the Weyl group (see [2]).

Lemma 5 (This lemma complete the proof of Theorem 1). *H is isomorphic to either* $0(2n, 2, +1)$ *or* $0(2n, 2, -1)$.

Proof of Lemma 5. Let H be the subgroup of H generated by all elations in H. From Lemma 4, we can see that H_0 contains $2^{n-1}(2^n-1)$ or $2^{n-1}(2^n+1)$ elations according as $\chi=\chi_1$ or χ_2. Using this fact we can prove first that H is an irreducible subgroup, and next that H_0 is an irreducible subgroup. The final step of the identification is done using the classification theorem of irreducible subgroups of $SL(2n, 2)$ generated by elations (transvections) due to J. McLaughlin [5].

References

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