27. A Remark on the Approximate Spectra of Operators

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1. In the present note, several equivalent conditions on the approximate spectrum of an operators will be discussed in § 2. The joint approximate spectrum introduced by Bunce [5] is also discussed in § 4. In § 3, an algebraic proof of Wintner-Hildebrandt-Orland's theorem is given.

2. The equivalence of several definitions on an approximate propervalue is unified in the following theorem:

Theorem 1. *For an operator T on a Hilbert space $\mathfrak{H}$, the following conditions are equivalent*:

(i) *For any $\varepsilon>0$, there is a vector $x\in\mathfrak{H}$ with $\|x\|=1$ and*

$$\|Tx-\lambda x\|<\varepsilon, \tag{1}$$

(ii) *There is a sequence of operators S_n with $\|S_n\|=1$ and*

$$\|(T-\lambda)S_n\|\to 0 \qquad (n\to\infty), \tag{2}$$

(iii) *Let $\mathfrak{B}(\mathfrak{H})$ be the algebra of all operators, then*

$$\mathfrak{B}(\mathfrak{H})(T-\lambda)\neq\mathfrak{B}(\mathfrak{H}), \tag{3}$$

(iv) *There is no $\varepsilon>0$ such that*

$$(T-\lambda)^*(T-\lambda)\geqq\varepsilon. \tag{4}$$

Historically, (i) is the original definition of Halmos [7; p. 51], (ii) is due to Berberian [1; VII, § 3, Ex. 10], (iii) is introduced very recently by Bunce [4] and (iv) is pointed out by Berberian [2].

If λ satisfies one of the above conditions, λ will be called an *approximate propervalue* of T. The set $\pi(T)$ of all approximate propervalues of T is called the *approximate spectrum* of T.

(i) implies (ii): This is already contained in [1]. Suppose

$$\|Tx_n-\lambda x_n\|\to 0 \qquad (n\to\infty)$$

for $\|x_n\|=1$. If $S_n=x_n\otimes x_n$ in the sense of Schatten [11], i.e.

$$(y\otimes z)x=(x\,|\,z)y,$$

then S_n is a one-dimensional projection, so that

$$\|S_n\|=1, \quad \|(T-\lambda)S_n\|\to 0 \quad (n\to\infty).$$

(ii) implies (iii): $T-\lambda$ is a right generalized divisor of zero [10; p. 27]; hence $\mathfrak{B}(\mathfrak{H})(T-\lambda)$ consists of generalized divisors of zero which implies (iii).

(iii) implies (iv): If not,

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$$(4)' \qquad (T-\lambda)^*(T-\lambda) \geqq \varepsilon > 0;$$

hence $T-\lambda$ is left-invertible, so that

$$(3)' \qquad \mathfrak{B}(\mathfrak{H})(T-\lambda) = \mathfrak{B}(\mathfrak{H}).$$

(iv) implies (i): (4) implies that there is a projection P such that

$$\|(T-\lambda)^*(T-\lambda)P\| < \varepsilon.$$

If $x \in \operatorname{ran} P$, then

$$\|Tx - \lambda x\|^2 = ((T-\lambda)^*(T-\lambda)x \mid x) \leqq \|(T-\lambda)^*(T-\lambda)Px\| < \varepsilon$$

as desired.

There are similar another equivalent conditions instead of (i)–(iv). For example,

(ii′) *There are projections* P_n *such that* $\|(T-\lambda)P_n\| \to 0$.

Clearly(ii′) is equivalent to (ii) as observed in the above. Also

(iii′) *If* $\mathfrak{A}$ *is a unital* C^**-algebra containing* T, *then*

$$(3)' \qquad \mathfrak{A}(T-\lambda) \neq \mathfrak{A}.$$

(iii′) is equivalent to (iii); since (iv) shows that the approximate spectrum of an operator is purely algebraical. The equivalence of (iii) and (iii′) is already observed by Bunce [4].

3. A typical example of approximate spectra is given by the following theorem:

Theorem 2 (Wintner-Hildebrandt [8], Orland [9]). *If* $\lambda \in \bar{W}(T)$ *and* $|\lambda| = \|T\|$, *then* λ *is an approximate propervalue, where* $\bar{W}(T)$ *is the closure of the numerical range*

$$(5) \qquad W(T) = \{(Tx \mid x) \mid \|x\| = 1\}.$$

The original proof of [8] and [9] is simple. However, an algebraic proof is given based on the following theorem:

Theorem 3 (Berberian-Orland [3]). *If* $\mathfrak{A}$ *is a unital* C^**-algebra with the state space* Σ, *then*

$$(6) \qquad \bar{W}(T) = \Sigma(T) = \{\rho(T) \mid \rho \in \Sigma\}$$

for any $T \in \mathfrak{A}$.

For the proof of Theorem 2, it is obvious that one can assume $\lambda = 1$ and $\|T\| = 1$. If 1 is not an approximate propervalue, then T satisfies (4) for an $\varepsilon > 0$. Hence

$$T^*T + 1 \geqq \varepsilon + 2 \operatorname{Re} T,$$

where

$$\operatorname{Re} T = \frac{T+T^*}{2}.$$

By (6), there is $\rho \in \Sigma$ such as $\rho(T) = 1$. Therefore

$$2 \geqq \rho(T^*T) + 1 \geqq \varepsilon + 2 > 2,$$

which is a contradiction.

Incidentally, in the remainder of this section, an alternative proof of Theorem 3 will be given, which is essentially due to Z. Takeda.

Since a state of a C^*-algebra acting on $\mathfrak{H}$ is extendable to a state of $\mathfrak{B}(\mathfrak{H})$, it is not restrictive to assume that $\mathfrak{A}=\mathfrak{B}(\mathfrak{H})$. Let Σ' be the set of all vector states such as

(7) $$\rho(T)=(Tx\mid x) \qquad (\|x\|=1).$$

Clearly, Σ' satisfies

(8) $$W(T)=\Sigma'(T)=\{\rho(T)\mid \rho\in\Sigma'\}.$$

Let Σ'' be the norm convex closure of Σ'. Then $\Sigma''(T)\subset \bar{W}(T)$ and

$$W(T)=\Sigma'(T)\subset\Sigma''(T)\subset\Sigma(T).$$

By a theorem of Dixmier [6], it is known that Σ' consists of all strongestly continuous states of $\mathfrak{B}(\mathfrak{H})$. Hence, to prove the theorem, it needs to show that Σ'' is weakly* dense in Σ. However, this is essentially a theorem of Takeda [12] which states that Σ'' is a *basic* subset of Σ. Therefore, $W(T)$ is dense in $\Sigma(T)$.

4. Bunce [5] introduced recently the notion of the joint spectrum of commuting operators $T_1,\cdots,T_n$. A set of n complex number s $\lambda_1,\cdots,\lambda_n$ is a *joint approximate propervalue* of $T_1,\cdots,T_n$ if

(9) $$\mathfrak{B}(\mathfrak{H})(T_1-\lambda_1)+\cdots+\mathfrak{B}(\mathfrak{H})(T_n-\lambda_n)\neq\mathfrak{B}(\mathfrak{H}).$$

The set $\pi(T_1,\cdots,T_n)$ of all joint approximate propervalues is called the *joint approximate spectrum* of $T_1,\cdots,T_n$. He proved, among others, $\pi(T_1,\cdots,T_n)$ is a non-void compact set which satisfies

(10) $$\pi(T_1,\cdots,T_n)\subset\pi(T_1)\times\cdots\times\pi(T_n).$$

Since Bunce's definition corresponds to (iii) in the case of a single operator, it is natural to ask that a definition corresponding to (iv) gives the the same spectrum.

Let $\pi'(T_1,\cdots,T_n)$ be the set of all n numbers which satisfy that there is no $\varepsilon>0$ such that

(11) $$(T_1-\lambda_1)^*(T_1-\lambda_1)+\cdots+(T_n-\lambda_n)^*(T_n-\lambda_n)\geqq\varepsilon.$$

It is clear that π' satisfies (10) too. Each point in the complement of $\pi'(T_1,\cdots,T_n)$ satisfies

(11)′ $$\sum_{i=1}^{n}(T_i-\lambda_i)^*(T_i-\lambda_i) \quad \textit{is invertible.}$$

Therefore, $\pi'(T_1,\cdots,T_n)$ is compact.

Theorem 4. *Two definitions of the joint approximate spectrum are equivalent*:

(12) $$\pi(T_1,\cdots,T_n)=\pi'(T_1,\cdots,T_n).$$

Instead of the equivalence of (9) and (11), the equivalence of the following two conditions will be proved:

(13) $$\mathfrak{B}(\mathfrak{H})(T_1-\lambda_1)+\cdots+\mathfrak{B}(\mathfrak{H})(T_n-\lambda_n)=\mathfrak{B}(\mathfrak{H})$$

and

(14) $$(T_1-\lambda_1)^*(T_1-\lambda_1)+\cdots+(T_n-\lambda_n)^*(T_n-\lambda_n)\geqq\varepsilon>0.$$

(14) implies (13): The hypothesis implies that there is a B such as

$$B\sum_{i=1}^{n}(T_i-\lambda_i)^*(T_i-\lambda_i)=1.$$

Hence for every $C\in\mathfrak{B}(\mathfrak{H})$,

$$CB\sum_{i=1}^{n}(T_i-\lambda_i)^*(T_i-\lambda_i)=C,$$

so that (13) is satisfied.

(13) implies (14): If there are $B_1, \cdots, B_n$ such that

$$\sum_{i=1}^{n}B_i(T_i-\lambda_i)=1,$$

then for any vector x

$$\|x\|\leqq\sum_{i=1}^{n}\|B_i(T_i-\lambda_i)x\|\leqq\sum_{i=1}^{n}\|B_i\|\,\|T_ix-\lambda_ix\|$$
$$\leqq m\sum_{i=1}^{n}\|(T_i-\lambda_i)x\|$$

where

$$m=\max(\|B_1\|,\cdots,\|B_n\|),$$

and so

$$\frac{\|x\|}{m}\leqq\sum_{i=1}^{n}\|(T_i-\lambda_i)x\|.$$

Therefore

$$\frac{\|x\|^2}{m}\leqq\left[\sum_{i=1}^{n}\|(T_i-\lambda_i)x\|\right]^2\leqq n\sum_{i=1}^{n}\|(T_i-\lambda_i)x\|^2$$
$$=n\left(\sum_{i=1}^{n}(T_i-\lambda_i)^*(T_i-\lambda_i)x\,|\,x\right).$$

Hence

$$\sum_{i=1}^{n}(T_i-\lambda_i)^*(T_i-\lambda_i)\geqq\frac{1}{nm}>0,$$

as desired.

The joint spectrum of elements of a commutative Banach algebra is introduced by Arens and Calderon, cf. [10; p. 150]. Bunce [5] established that the joint approximate spectrum of commuting hyponormal operators is included in the joint spectrum in the sense of Arens-Calderon. However, for general operators, there is no further information.

Naturally, there is an another definition of the approximate spectrum of operators is possible. Corresponding to (ii), $(\lambda_1, \cdots, \lambda_n)$ belongs to the joint approximate spectrum if there is a sequence of projections P_n such that

(15) $$\|(T_i-\lambda_i)P_k\|\to 0 \quad (k\to\infty) \quad (i=1,2,\cdots,n).$$

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