26. *On Axioms of Boolean Algebra*

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An axiom system of the implicational calculus is given in the form of an algebra $M=\langle X, 0, *\rangle$ satisfying the following conditions:

1) $x*y\leqslant x$,
2) $(x*y)*(x*z)\leqslant z*y$,
3) $x\leqslant x*(y*x)$,
4) $0\leqslant x$,
5) $x*y=0$ if and only if $x\leqslant y$.

In our Note [1], the algebra M is called an I-algebra. In this algebra M, we shall introduce a new element 1 called unit element satisfying $x\leqslant 1$ for every element x of X. If we define $\sim x=1*x$, we have a Boolean algebra.

Theorem 1. *An I-algebra M with unit 1 satisfying $x\leqslant 1$ for every $x\in X$ is a Boolean algebra.*

In the proof of Theorem, we shall use some results in [1] without proofs. If we verify the following conditions:

1) $(x*y)*(x*z)\leqslant z*y$,
2) $y*(1*x)\leqslant x$,
3) $x\leqslant x*(1*x)$,

then M is a Boolean algebra with complementation $\sim x$ defined by $\sim x=1*x$.

Proof. The first condition is the second axiom of the I-algebra. To prove $y*(1*x)\leqslant x$, we shall show $(y*(1*x))*x=0$.

$$\begin{aligned}(y*(1*x))*x&=(y*x)*(1*x) && ((9)\text{ in }[1])\\ &\leqslant(y*1)*x && ((14)\text{ in }[1])\\ &=(y*x)*1 && ((9)\text{ in }[1])\\ &=0.\end{aligned}$$

The third condition is obtained from $x\leqslant x*(y*x)$. Take y as 1, then we have $x\leqslant x*(1*x)$. Therefore we complete the proof of Theorem 1.

By Theorem 1 and some results mentioned in our Note [2]-[4] and [5], we have the following characterizations of the Boolean algebra with unit.

Let $\langle X, 0, 1, *\rangle$ be an algebra with zero 0 and unit 1, where $*$ is a binary operation on X.

Theorem 2. *The algebra $\langle X, 0, 1, *\rangle$ is a Boolean algebra with unit, if it satisfies the following conditions:*

1) $v \leqslant (v*(((u*r)*(u*s))*(u*(t*(s*r))))*((p*q)*p)$,
2) $0 \leqslant x \leqslant 1$,
3) $x*y=0$ *if and only if* $x \leqslant y$.

Theorem 3. *The algebra* $\langle X, 0, 1, *\rangle$ *is a Boolean algebra with unit, if it satisfies the following conditions*:
1) $((t*s)*(u*p))*((t*(s*r))*u) \leqslant (t*(s*r))*(q*p)$,
2) $0 \leqslant x \leqslant 1$,
3) $x*y=0$ *if and only if* $x \leqslant y$.

Theorem 4. *The algebra* $\langle X, 0, 1, *\rangle$ *is a Boolean algebra with unit, if it satisfies the following conditions*:
1) $(p*q) \leqslant p$,
2) $(s*p)*(s*q) \leqslant s*(r*(q*p))$,
3) $0 \leqslant x \leqslant 1$,
4) $x*y=0$ *if and only if* $x \leqslant y$.

Theorem 5. *The algebra* $\langle X, 0, 1, *\rangle$ *is a Boolean algebra with unit, if it satisfies the following conditions*:
1) $(s*p)*(s*q) \leqslant q*p$,
2) $p \leqslant p*(q*p)$,
3) $q*(q*p) \leqslant p$,
4) $0 \leqslant x \leqslant 1$,
5) $x*y=0$ *if and only if* $x \leqslant y$.

Theorem 6. *The algebra* $\langle X, 0, 1, *\rangle$ *is a Boolean algebra, if it satisfies the following conditions*:
1) $x \leqslant x*(y*x)$,
2) $(x*y)*(z*u) \leqslant x*(z*(u*y))$,
3) $0 \leqslant x \leqslant 1$,
4) $x*y=0$ *if and only if* $x \leqslant y$.

In each theorem, we suppose $x=y$ is defined by $x \leqslant y$ and $y \leqslant x$.

References

[1] Y. Imai and K. Iséki: On axiom system of propositional calculi. XIV. Proc. Japan Acad., **42**, 19–22 (1966).
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[5] ——: On axiom systems of propositional calculi. XXV. Proc. Japan Acad., **43**, 192–193 (1967).