16. *On the Uniform Convergence of a Finite Difference Scheme for a Nonlinear Heat Equation*

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1. Introduction. Let Ω be a bounded domain in R^d with smooth boundary $\partial\Omega$ and let $C_0(\Omega)$ be the Banach space of all continuous functions f on $\bar{\Omega}$ satisfying $f(x)=0$ for $x\in\partial\Omega$, with the norm $\|f\| =\max_{x\in\Omega}|f(x)|$. We set $a\in C_0(\Omega)$. In the present paper, we consider the differential equation

$$\varphi(\partial u/\partial t)-\Delta u+\gamma(u)=0 \qquad \text{in } \Omega\times(0,\infty) \tag{1.1}$$

with the boundary condition

$$u(x,t)=0 \qquad \text{on } \partial\Omega\times(0,\infty) \tag{1.2}$$

and the initial condition

$$u(x,0)=a(x) \qquad \text{in } \Omega, \tag{1.3}$$

where (and throughout the present paper unless otherwise stated) $\varphi =\varphi(r)$ is a strictly monotone increasing continuous function defined on R^1 satisfying $\lim_{r\uparrow\infty}\varphi(r)=\infty$, $\lim_{r\downarrow-\infty}\varphi(r)=-\infty$ and $\varphi(0)=0$, Δ is the Laplace operator in the space variable x and $\gamma=\gamma(r)$ is a monotone non-decreasing continuous function defined on R^1 satisfying $\gamma(0)=0$.

Let h and k be positive numbers and define the following implicit finite difference scheme (1.4) which is an analogue of the problem (1.1)–(1.2)–(1.3):

$$\begin{cases}\varphi((u^n_{i_1,i_2,\cdots,i_d}-u^{n-1}_{i_1,i_2,\cdots,i_d})/k)-\Delta_{(h)}u^n_{i_1,i_2,\cdots,i_d}+\gamma(u^n_{i_1,i_2,\cdots,i_d})=0,\\ \qquad i_1,i_2,\cdots,i_d \text{ integers}, (i_1h,i_2h,\cdots,i_dh)\in\Omega, n=1,2,\cdots,\\ u^0_{i_1,i_2,\cdots,i_d}=a(i_1h,i_2h,\cdots,i_dh), (i_1h,i_2h,\cdots,i_dh)\in\Omega,\end{cases} \tag{1.4}$$

where

$$\Delta_{(h)}\xi_{i_1,i_2,\cdots,i_d}=\sum_{j=1}^{d}\Delta_{(h),j}\xi_{i_1,i_2,\cdots,i_d} \tag{1.5}$$

and each term $\Delta_{(h),j}\xi_{i_1,i_2,\cdots,i_d}$ in the right-hand side of the above formula is defined as follows.

Case 1. If $(i_1h,\cdots,i_{j-1}h,(i_j\pm1)h,i_{j+1}h,\cdots,i_dh)\in\Omega$, then

$$\Delta_{(h),j}\xi_{i_1,i_2,\cdots,i_d}=(\xi_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_d}-2\xi_{i_1,i_2,\cdots,i_d}+\xi_{i_1,\cdots,i_{j-1},i_j-1,i_{j+1},\cdots,i_d})/h^2.$$

Case 2. If $(i_1h,\cdots,i_{j-1}h,(i_j+1)h,i_{j+1}h,\cdots,i_dh)\notin\Omega$ and $(i_1h,\cdots, i_{j-1}h,(i_j-1)h,i_{j+1}h,\cdots,i_dh)\in\Omega$, then

$$\Delta_{(h),j}\xi_{i_1,i_2,\cdots,i_d}=2\{(\theta_{i_1,i_2,\cdots,i_d;j}+1)^{-1}\xi_{i_1,\cdots,i_{j-1},i_j-1,i_{j+1},\cdots,i_d}-\theta^{-1}_{i_1,i_2,\cdots,i_d;j}\xi_{i_1,i_2,\cdots,i_d}\}/h^2,$$

where

$\theta_{i_1,i_2,\cdots,i_d;j}=\inf\{\theta\in(0,1];(i_1h,\cdots,i_{j-1}h,(i_j+\theta)h,i_{j+1}h,\cdots,i_dh)\notin\Omega\}>0.$

Case 3. If $(i_1h,\cdots,i_{j-1}h,(i_j+1)h,i_{j+1}h,\cdots,i_dh)\in\Omega$ and $(i_1h,\cdots,i_{j-1}h,(i_j-1)h,i_{j+1}h,\cdots,i_dh)\notin\Omega$, then

$$\Delta_{(h),j}\xi_{i_1,i_2,\cdots,i_d}=2\{(\bar{\theta}_{i_1,i_2,\cdots,i_d;j}+1)^{-1}\xi_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_d}-\bar{\theta}^{-1}_{i_1,i_2,\cdots,i_d;j}\xi_{i_1,i_2,\cdots,i_d}\}/h^2,$$

where

$\bar{\theta}_{i_1,i_2,\cdots,i_d;j}=\inf\{\theta\in(0,1];(i_1h,\cdots,i_{j-1}h,(i_j-\theta)h,i_{j+1}h,\cdots,i_dh)\notin\Omega\}>0.$

Case 4. If $(i_1h,\cdots,i_{j-1}h,(i_j\pm1)h,i_{j+1}h,\cdots,i_dh)\notin\Omega$, then

$$\Delta_{(h),j}\xi_{i_1,i_2,\cdots,i_d}=-2\xi_{i_1,i_2,\cdots,i_d}\theta^{-1}_{i_1,i_2,\cdots,i_d;j}\cdot\bar{\theta}^{-1}_{i_1,i_2,\cdots,i_d;j}/h^2,$$

where $\theta_{i_1,i_2,\cdots,i_d;j}$ and $\bar{\theta}_{i_1,i_2,\cdots,i_d;j}$ are as defined above.

Our purpose is to prove the following theorem by the recent *theory of nonlinear semi-groups.*

Theorem 1. (i) *There exists a unique solution* $\{u^n_{i_1,i_2,\cdots,i_d}\}$ *of* (1.4).

(ii) *Fix an arbitrary positive number T. Under the additional assumption that* φ *and* γ *are continuously differentiable, we have*

(1.6) $$\lim_{h\downarrow0,k\downarrow0}\ \sup_{\substack{n\le T/k\\(i_1h,i_2h,\cdots,i_dh)\in\Omega}}|u^n_{i_1,i_2,\cdots,i_d}-u((i_1h,i_2h,\cdots,i_dh),nk)|=0$$

for some $u=u(x,t)\in C([0,T];C_0(\Omega))$.

Remark 1. u in Theorem 1 will be given by a (*nonlinear*) *contraction semi-group* $\{S_t\}_{t\ge0}$ in $C_0(\Omega)$:

(1.7) $$u(\cdot,t)=S_ta,\qquad 0\le t\le T.$$

Such a semi-group will be constructed in § 2.

Remark 2. In the case $\gamma(r)\equiv0$, (1.1) is formally equivalent to the *nonlinear heat equation*: $c\rho(\partial v/\partial t)=\mathrm{div}\,(\kappa\,\mathrm{grad}\,v)$, where specific heat c, density ρ and heat conductivity κ depend on the temperature $v=v(x,t)$; $K'=\kappa$, $\varphi'\circ K=c\rho/\kappa$, $\partial u/\partial t=K\circ v$. Concerning this, see [10], p. 49 (cf. [9]).

2. Preliminaries.[1] By definition a (possibly) nonlinear operator A in real Banach space X is *dissipative* if $\|(f-\lambda Af)-(g-\lambda Ag)\|\ge\|f-g\|$ whenever $f,g\in D(A)$ for each $\lambda>0$. The dissipativity of A is equivalent to the condition: $\tau(f-g,-Af+Ag)\ge0$ whenever $f,g\in D(A)$, where $\tau(f,g)=\lim_{\varepsilon\downarrow0}\varepsilon^{-1}(\|f+\varepsilon g\|-\|f\|)$, $f,g\in X$. A dissipative operator A in X is said to be *m-dissipative* if $R(I-\lambda A)=X$ for every, or equivalently, for some $\lambda>0$.

Let Λ be the infinitesimal generator of a "compact"[2] contraction semi-group $\{\exp(t\Lambda);t\ge0\}$ of class (C_0) in $C_0(\Omega)$. For φ introduced in § 1, we set $\varphi^{-1}=\beta$ and define the operator $\bar{\beta}$ in $C_0(\Omega)$ by

(2.1) $D(\bar{\beta})=C_0(\Omega)$, $(\bar{\beta}f)(x)=\beta(f(x))$, $x\in\Omega$, for $f\in D(\beta)$.

Similarly one defines the operator $\bar{\gamma}$ in $C_0(\Omega)$.

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1) In this section we discuss in the abstract setting.

2) A semi-group $\{\exp(t\Omega);t\ge0\}$ of class (C_0) in Banach space is said to be *compact* if $\exp(t\Omega)$ is compact for every $t>0$ (see [7]).

Proposition 2. *The product $\bar{\beta}(\Lambda-\bar{\gamma})$ of $\bar{\beta}$ and $\Lambda-\bar{\gamma}$ is an m-dissipative operator with domain dense in $C_0(\Omega)$.*

In the case $X=C_0(\Omega)$,

$$\tau(f,g)=\max_{x\in\{x;|f(x)|=\|f\|\}}(\operatorname{sgn} f(x))g(x),\qquad f,g\in C_0(\Omega),\ f\neq 0 \tag{2.2}$$

([8], § 6). Consequently we have

Lemma 3. *The product $\bar{\beta}A$ of $\bar{\beta}$ and a dissipative operator A in $C_0(\Omega)$ is dissipative.*

Proof of Proposition 2. Since $\Lambda-\bar{\gamma}$ is dissipative, $\bar{\beta}(\Lambda-\bar{\gamma})$ is dissipative by Lemma 3. We shall prove the relation $R(I-\bar{\beta}(\Lambda-\bar{\gamma}))=C_0(\Omega)$. We introduce the *Yosida approximation* $\Lambda_\varepsilon(\varepsilon>0)$ of Λ, as usual, by $\Lambda_\varepsilon=\varepsilon^{-1}\{(I-\varepsilon\Lambda)^{-1}-I\}$ $(=\Lambda(I-\varepsilon\Lambda)^{-1})$, which is a continuous dissipative operator defined on $C_0(\Omega)$. Since $\bar{\beta}(\Lambda_\varepsilon-\bar{\gamma})$ $(\varepsilon>0)$ is a continuous dissipative operator defined on $C_0(\Omega)$, it is m-dissipative (see [5]). Accordingly, for an arbitrarily fixed $w\in C_0(\Omega)$, there exists $f_\varepsilon\in C_0(\Omega)$ satisfying

$$f_\varepsilon-\bar{\beta}(\Lambda_\varepsilon-\bar{\gamma})f_\varepsilon=w \tag{2.3}$$

for each $\varepsilon>0$. By the dissipativity of $\bar{\beta}(\Lambda_\varepsilon-\bar{\gamma})$ we have

$$\|f_\varepsilon\|\leq\|w\| \tag{2.4}$$

for each $\varepsilon>0$. Noticing that

$$\begin{aligned}\|(I-\Lambda)(I-\varepsilon\Lambda)^{-1}f_\varepsilon\|&\leq\|(I-\varepsilon\Lambda)^{-1}f_\varepsilon\|+\|\bar{\beta}^{-1}(f_\varepsilon-w)\|+\|\bar{\gamma}f_\varepsilon\|\\&\leq\|w\|+\max(\varphi(2\|w\|),-\varphi(-2\|w\|))+\max(\gamma(\|w\|),-\gamma(-\|w\|))\end{aligned} \tag{2.5}$$

and that $(I-\Lambda)^{-1}$ is compact (see Theorem 3.3 in [7]), we can conclude that the set $\{(I-\varepsilon\Lambda)^{-1}f_\varepsilon;\ \varepsilon>0\}$ is relatively compact in $C_0(\Omega)$. Consequently there exists a sequence $\varepsilon_n\downarrow 0$ $(\varepsilon_n>0)$ such that

$$\operatorname*{s-lim}_{n\to\infty}(I-\varepsilon_n\Lambda)^{-1}f_{\varepsilon_n}=f \tag{2.6}$$

exists in $C_0(\Omega)$. From this, (2.6) and (2.3) one obtains

$$\begin{aligned}\overline{\lim_{n\to\infty}}\|f_{\varepsilon_n}-f\|&\leq\overline{\lim_{n\to\infty}}\|f_{\varepsilon_n}-(I-\varepsilon_n\Lambda)^{-1}f_{\varepsilon_n}\|=\overline{\lim_{n\to\infty}}\,\varepsilon_n\|\Lambda_{\varepsilon_n}f_{\varepsilon_n}\|\\&\leq\{\max(\varphi(2\|w\|),-\varphi(-2\|w\|))\\&\qquad+\max(\gamma(\|w\|),-\gamma(-\|w\|))\}\cdot\overline{\lim_{n\to\infty}}\,\varepsilon_n=0.\end{aligned} \tag{2.7}$$

Accordingly

$$\begin{aligned}&\operatorname*{s-lim}_{n\to\infty}\Lambda(I-\varepsilon_n\Lambda)^{-1}f_{\varepsilon_n}\\&=\operatorname*{s-lim}_{n\to\infty}(\bar{\beta}^{-1}(f_{\varepsilon_n}-w)+\bar{\gamma}f_{\varepsilon_n})=\bar{\beta}^{-1}(f-w)+\bar{\gamma}f.\end{aligned} \tag{2.8}$$

In view of (2.6) and (2.8) and by the closedness of Λ, $f\in D(\Lambda)$ and $\Lambda f=\bar{\beta}^{-1}(f-w)+\bar{\gamma}f$. Q.E.D.

Applying Theorem I in [2] to the operator $\bar{\beta}(\Lambda-\bar{\gamma})$, we obtain

Corollary 4. *$\bar{\beta}(\Lambda-\bar{\gamma})$ "generates" a (nonlinear) contraction semigroup on $C_0(\Omega)$:*

$$\exp(t\bar{\beta}(\Lambda-\bar{\gamma}))\cdot a=\operatorname*{s-lim}_{\lambda\downarrow 0}\{I-\lambda\bar{\beta}(\Lambda-\bar{\gamma})\}^{-[t/\lambda]}a \tag{2.9}$$

exists for $a\in C_0(\Omega)$, $t\geq 0$ and $\{\exp(t\bar{\beta}(\Lambda-\bar{\gamma}));\ t\geq 0\}$ belongs to $Q_0(C_0(\Omega))$ in the sense of [2].

Remark 3. Under the additional assumption that $\{\exp(t\Delta); t\geq 0\}$ is "non-negative" (see [8]), $\{\exp(t\bar{\beta}(\Delta-\bar{\gamma})); t\geq 0\}$ belongs to $Q_0^+(C_0(\Omega))$ in the sense of [3].

3. Proof of Theorem 1. Let $l^\infty_{(h)}$ be the finite dimensional Banach space of all real vectors $\{\xi_{i_1,i_2,\cdots,i_d}\}_{(i_1h,i_2h,\cdots,i_dh)\in\Omega}$, normed by $\|\{\xi_{i_1,i_2,\cdots,i_d}\}\|_{(h)} = \max_{(i_1h,i_2h,\cdots,i_dh)\in\Omega}|\xi_{i_1,i_2,\cdots,i_d}|$. Since, in the case $X=l^\infty_{(h)}$, the functional $\tau=\tau_{(h)}$ is of the form: $\tau_{(h)}(\{\xi_{i_1,i_2,\cdots,i_d}\},\{\eta_{i_1,i_2,\cdots,i_d}\})=\max(\operatorname{sgn}\xi_{i_1,i_2,\cdots,i_d})\cdot\eta_{i_1,i_2,\cdots,i_d}$, where the maximum is taken for all $(i_1,i_2,\cdots,i_d)$ satisfying $|\xi_{i_1,i_2,\cdots,i_d}|=\|\{\xi_{i_1,i_2,\cdots,i_d}\}\|_{(h)}$ ([8], § 6, cf. (2.2)), $\Delta_{(h)}$ defined in § 1 is dissipative in $l^\infty_{(h)}$. On the other hand, for β and γ we can define as in (2.1) the corresponding operators in $l^\infty_{(h)}$, which we denote also by $\bar{\beta}$ and $\bar{\gamma}$ respectively. Since $\bar{\beta}(\Delta_{(h)}-\bar{\gamma})$ is a continuous dissipative operator defined on $l^\infty_{(h)}$, it is m-dissipative ([5]), which proves Theorem 1, (i). We set

$$\exp(t\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))\cdot\{\xi_{i_1,i_2,\cdots,i_d}\}=\operatorname*{s-lim}_{\lambda\downarrow 0}(I-\lambda\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-[t/\lambda]}\{\xi_{i_1,i_2,\cdots,i_d}\}$$

for $\{\xi_{i_1,i_2,\cdots,i_d}\}\in l^\infty_{(h)}$.

Now define the linear contraction operator $P_{(h)}$ $(h>0)$ of $C_0(\Omega)$ into $l^\infty_{(h)}$ by $(P_{(h)}f)_{i_1,i_2,\cdots,i_d}=f(i_1h,i_2h,\cdots,i_dh),(i_1h,i_2h,\cdots,i_dh)\in\Omega$, for $f\in C_0(\Omega)$. We define the operator Δ_0 in $C_0(\Omega)$ by $D(\Delta_0)=\{f\in C_0(\Omega); f\in W^{2,q}(\Omega)$ and $\Delta f\in C_0(\Omega)\}$, $(d<q<\infty)$, $(\Delta_0 f)(x)=\Delta f(x)$, $f\in D(\Delta_0)$, which is independent of the choice of q. Thus Δ_0 is the infinitesimal generator of a compact contraction semi-group of class (C_0) in $C_0(\Omega)$ (see [6]).

Lemma 5. *Assume that φ and γ are continuously differentiable and fix an arbitrary positive number T. Then*

(3.1) $$\lim_{h\downarrow 0}\sup_{0\leq t\leq T}\|P_{(h)}\exp(t\bar{\beta}(\Delta_0-\bar{\gamma}))\cdot a-\exp(t\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))\cdot P_{(h)}a\|_{(h)}=0$$

for each $a\in C_0(\Omega)$.

Sketch of the proof of Lemma 5. Set $g\in C_0(\Omega)\cap C^1(\bar{\Omega})$. Noticing that $\varphi=\beta^{-1}$ and γ are continuously differentiable, we have

$$\tilde{g}_\lambda=(I-\lambda\bar{\beta}(\Delta_0-\bar{\gamma}))^{-1}g\in C_0(\Omega)\cap C^2(\bar{\Omega})$$

whenever $\lambda>0$. Consequently we have, remembering the definition of $\Delta_{(h)}$, that

$$\begin{aligned}&\overline{\lim_{h\downarrow 0}}\|P_{(h)}(I-\lambda\bar{\beta}(\Delta_0-\bar{\gamma}))^{-1}g-(I-\lambda\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-1}P_{(h)}g\|_{(h)}\\ &\leq\overline{\lim_{h\downarrow 0}}\|(I-\lambda\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))P_{(h)}\tilde{g}_\lambda-P_{(h)}(I-\lambda\bar{\beta}(\Delta_0-\gamma))\tilde{g}_\lambda\|_{(h)}\\ &=\lambda\overline{\lim_{h\downarrow 0}}\|\bar{\beta}(\Delta_{(h)}P_{(h)}\tilde{g}_\lambda-\bar{\gamma}P_{(h)}\tilde{g}_\lambda)-\bar{\beta}(P_{(h)}\Delta_0\tilde{g}_\lambda-\bar{\gamma}P_{(h)}\tilde{g}_\lambda)\|_{(h)}=0.\end{aligned}$$

Hence we have

$$\lim_{h\downarrow 0}\|P_{(h)}(I-\lambda\bar{\beta}(\Delta_0-\bar{\gamma}))^{-1}f-(I-\lambda\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-1}P_{(h)}f\|_{(h)}=0$$

for each $f\in C_0(\Omega)$ and $\lambda>0$, by means of which we can prove (3.1) (cf. Theorem 3.1 in [1]).

Proof of Theorem 1, (ii). First we assume that $a\in C_0(\Omega)\cap C^2(\bar{\Omega})$. We have

$$\|(I-k\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-n}P_{(h)}a-P_{(h)}\exp(nk\bar{\beta}(\Delta_0-\bar{\gamma}))\cdot a\|_{(h)}$$
$$\leq\|(I-k\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-n}P_{(h)}a-\exp(nk\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))\cdot P_{(h)}a\|_{(h)}$$
$$+\|\exp(nk\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))\cdot P_{(h)}a-P_{(h)}\exp(nk\bar{\beta}(\Delta_0-\bar{\gamma}))\cdot a\|_{(h)}$$

and, by the estimate (1.10) in [2],

$$\|(I-k\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-n}P_{(h)}a-\exp(nk\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))\cdot P_{(h)}a\|_{(h)}$$
$$\leq 2k\sqrt{n}\,\|\bar{\beta}(\Delta_{(h)}P_{(h)}a-\bar{\gamma}P_{(h)}a)\|_{(h)}.$$

Accordingly by Lemma 5 we obtain

$$\lim_{h\downarrow 0,k\downarrow 0}\sup_{n\leq T/k}\|(I-k\bar{\beta}(\Delta_{(h)}-\bar{\gamma}))^{-n}P_{(h)}a-P_{(h)}\exp(nk\bar{\beta}(\Delta_0-\bar{\gamma}))\cdot a\|_{(h)}=0.$$

It is easy to prove the above equality for $a\in C_0(\Omega)$, which is nothing but (1.6) with

(1.7)′ $$u(\cdot,t)=\exp(t\bar{\beta}(\Delta_0-\bar{\gamma}))\cdot a;$$

this is the explicit form of (1.7). Q.E.D.

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