14. A Note on Schütte's Interpolation Theorem

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In this note, we shall add some remarks on Schütte's interpolation theorem in the intuitionistic predicate logic (cf. Schütte [3]), one of which give an affirmative solution of one of open problems in Gabbay [1].

Schütte's interpolation theorem. *If $A \supset B$ is provable in the intuitionistic predicate logic, then there is a formula C satisfying the following* (1) *and* (2):

(1) *$A \supset C$ and $C \supset B$ are provable in this logic.*

(2) *Every predicate symbol in C occurs both in A and in B.*

We add the following fact to this theorem:

Theorem. *In Schütte's theorem above, if A and B are built up using $\neg$ (negation), $\wedge$ (conjunction) and $\forall$ (universal quantification) only, then we can take such a C which satisfies* (1), (2) *and an added condition* (3):

(3) *Every free variable in C occurs both in A and in B.*

Remark 1. *The proposition obtained from the above theorem by omitting* (3) *is an affirmative solution of one of open problems in* [1].

Remark 2. *In Schütte's theorem, we can easily add the condition* (3) *to C, but in our theorem this is not trivial because we can not apply $\exists$ (existential quantifier) to C.*

Let LJ be the intuitionistic predicate logic formulated by Gentzen in [2]. For the sake of simplicity we assume that a sequent in LJ is of the form $\Gamma \to \Theta$, where Γ and Θ are finite sets of formulas in LJ and Θ has at most one formula, although we shall write $A, \Gamma \to B$ instead of $\{A\} \cup \Gamma \to \{B\}$. Furthermore we assume that LJ has two propositional constants $\top$ (truth), $\perp$ (false) and two added axiom sequents $\to \top$ and $\perp \to$.

Lemma 1. *Let $\Gamma \to \Theta$ be a sequent in LJ and (Γ_1, Γ_2) be an ordered partition of Γ. If $\vdash_{LJ} \Gamma \to \Theta$, then there is a formula C such that*

(4) *$\vdash_{LJ} \Gamma_1 \to C$ and $\vdash_{LJ} C, \Gamma_2 \to \Theta$.*

(5) *Every predicate symbol in C occurs both in Γ_1 and $\Gamma_2 \cup \Theta$. Furthermore if every formula in $\Gamma \cup \Theta$ is built up using $\neg, \wedge, \forall$ only, then C is also such a formula.*

Proof. We use the induction on a cut-free derivation $\mathcal{D}$ of $\Gamma \to \Theta$. We only treat the case that the last rule of $\mathcal{D}$ is $(\neg \to)$ or $(\to \forall)$.

Case 1. The last rule of $\mathcal{D}$ is $(\neg \to)$. Then $\mathcal{D}$ has the form

$$(\neg\rightarrow)\frac{\Gamma\overset{\downarrow}{\longrightarrow}A}{\neg A,\Gamma\rightarrow}.$$

If we divide $\neg A, \Gamma$ by $(\{\neg A\}\cup\Gamma_1, \Gamma_2)$, then by the hypothesis of induction there is a formula C_1 satisfying (4), (5) for the sequent $\Gamma\rightarrow A$ and the partition (Γ_2, Γ_1). Let $C=\neg C_1$.

If we divide $\neg A, \Gamma$ by $(\Gamma_1, \{\neg A\}\cup\Gamma_2)$, then by the hypothesis of induction, there is a formula C_1 satisfying (4), (5) for the sequent $\Gamma\rightarrow A$ and the partition (Γ_1, Γ_2). Let $C=C_1$.

Case 2. The last rule of $\mathscr{D}$ is $(\rightarrow\forall)$. Then $\mathscr{D}$ has the form

$$(\rightarrow\forall)\frac{\Gamma\overset{\downarrow}{\longrightarrow}A(a)}{\Gamma\rightarrow(\forall v)A(v)}, \quad \begin{array}{l} a \text{ does not occur in} \\ \text{the lower sequent.}\end{array}$$

Let (Γ_1, Γ_2) be an ordered partition of Γ. By the hypothesis of induction there is a formula $C_1(a)$ satisfying (4), (5) for the sequent $\Gamma\rightarrow A(a)$ and the partition (Γ_1, Γ_2). Let $C=(\forall v)C_1(v)$. Q.E.D.

Lemma 2. *If A and B are built up using $\neg, \wedge, \forall$ only and $\vdash_{LJ} A\rightarrow B$, then there is a formula C such that*

(6) $\vdash_{LJ}A\supset C$ *and* $\vdash_{LJ}C\supset B$.

(7) *Every predicate symbol in C occurs in A.*

(8) *Every free variable in C occurs both in A and in B.*

(9) *C is built up using $\neg, \wedge, \forall$ only.*

Proof. By the induction on B.

Case 1. B is an atomic formula. If B is $\top$ or $\bot$, obvious. If $B=P(a_1, \cdots, a_n)$ and P does not occur in A, then let $C=\bot$. If $B=P(a_1, \cdots, a_n)$ and P occur in A, let C be the formula obtained from B by applying $\forall$ to every free variable in B which does not occur in A.

Case 2. B is $\neg B_1$. Since $\vdash_{LJ}A\rightarrow B$, we have $\vdash_{LJ}B_1, A\rightarrow$. Let $a_1, \cdots, a_n$ be the set of free variables in A which do not appear in B and $C=\neg(\forall v_1)\cdots(\forall v_n)\neg A(v_1, \cdots, v_n)$, where $A=A(a_1, \cdots, a_n)$.

Case 3. B is $B_1\wedge B_2$. Since $\vdash_{LJ}A\rightarrow B_1\wedge B_2$ we have $\vdash_{LJ}A\rightarrow B_1$ and $\vdash_{LJ}A\rightarrow B_2$. By the hypotheses of induction, there are formulas C_1, C_2 satisfying (6)–(9) for $A\rightarrow B_1$ and $A\rightarrow B_2$. Let $C=C_1\wedge C_2$.

Case 4. B is $(\forall v)B_1(v)$. Let a be a free variable not in A, B. Since $\vdash_{LJ}A\rightarrow(\forall v)B_1(v)$, we have $\vdash_{LJ}A\rightarrow B_1(a)$. By the hypothesis of induction, there is a C_1 satisfying (6)—(9) for $A\rightarrow B_1(a)$. Let $C=C_1$. Q.E.D.

The proof of Schütte's theorem is obvious from Lemma 1. Assume that A and B are built up using $\neg, \wedge, \forall$ only and $\vdash_{LJ}A\rightarrow B$. Then by Lemma 1, there is such a formula C_1 satisfying (1), (2). By applying $\forall$, we can assume that every free variable in C_1 occurs in A. Then by using Lemma 2 to $\vdash_{LJ}C_1\rightarrow B$, there is a formula C satisfying (6)–(9) for the sequent $C_1\rightarrow B$.

Then clearly this C satisfies (1), (2) and (3).

Hence our theorem has been proved.

References

[1] D. M. Gabbay: Semantic proof of Craig's interpolation theorem for intuitionistic logic and extensions, Part II. Manchester Proc. (North-Holland Publ. Co.), 403–410 (1969).

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[3] K. Schütte: Der Interpolations-satz der intuitionistischen Pradikatenlogik. Math. Annalen, **148**, 192–200 (1962).