13. On Deformations of Holomorphic Maps

By Eiji HORIKAWA
University of Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., Feb. 12, 1972)

0. Introduction. The modern deformation theory has begun with the splendid work of Kodaira-Spencer [1] followed by [2], [3]. Moreover Kodaira has investigated families of submanifolds of a fixed compact complex manifold in [4]. The next natural problem is to investigate "deformations of holomorphic maps". I intend to give here a statement of fundamental results and some applications. Details will be published elsewhere.

1. Notations and conventions. We denote by X, Y, Z compact complex manifolds and by $p: \mathcal{X} \to M$, $q: \mathcal{Y} \to N$, $\pi: \mathcal{Z} \to S$ complex analytic families of compact complex manifolds (see [1] for the definition).

We say that two holomorphic maps $f: X \to Y$ and $f': X' \to Y$ are equivalent if there exists a complex analytic isomorphism $h: X \to X'$ such that $f = f' \circ h$.

2. Deformations of non-degenerate holomorphic maps. By a family of holomorphic maps into a fixed compact complex manifold Y, we mean a quadruplet $(\mathcal{X}, \Phi, p, M)$ of complex analytic family $p: \mathcal{X} \to M$ and a holomorphic map $\Phi: \mathcal{X} \to \mathcal{Y} = Y \times M$ over M in the sense that $p = pr_2 \circ \Phi$.

We define the concept of completeness of a family of holomorphic maps into Y as in the theory of deformations of compact complex manifolds [1].

Let $(\mathcal{X}, \Phi, p, M)$ be a family of holomorphic maps into Y, $0 \in M$, $X = X_0 = p^{-1}(0)$ and let $f = \Phi_0: X \to Y$ be the induced holomorphic map. Then we have an exact sequence of sheaves on X:

$$\Theta_X \xrightarrow{F} f^*\Theta_Y \xrightarrow{P} \mathcal{T} \longrightarrow 0$$

where θ denotes the sheaf of germs of holomorphic vector fields, $\mathcal{T} = \mathcal{T}_{X/Y}$ is the cokernel of the canonical homomorphism F and P is the natural projection.

For simplicity we assume that f is non-degenerate (i.e. $\operatorname{rank}_z df = \dim X$ for some point $z \in X$). Then the homomorphism F is injective. If f is an embedding, $\mathcal{T}$ is nothing but the normal bundle $\mathcal{N}$.

Now we define a characteristic map

$$\tau = \tau_0: T_0(M) \longrightarrow H^0(X, \mathcal{T})$$

($T_0(M)$ is the tangent space of M at 0) by the formula

$$\tau\left(\frac{\partial}{\partial t}\right)=P\left(\sum \frac{\partial \Phi^{\lambda}}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial w^{\lambda}}\right) \qquad \text{for } \frac{\partial}{\partial t} \in T_0(M)$$

(where $w=(w^1, \cdots, w^m)$ is a system of local coordinates on Y).

Theorem 1. *Let $(\mathcal{X}, \Phi, p, M)$ be a family of non-degenerate holomorphic maps into Y, $0 \in M, X=X_0$ and $f=\Phi_0: X \to Y$. If the characteristic map τ_0 is surjective, then the family is complete at* 0.

Theorem 2. *Let $f: X \to Y$ be a non-degenerate holomorphic map. If $H^1(X, \mathcal{T})=0$, then there exists a family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y and a point $0 \in M$ such that*

i) *$\Phi_0: X_0 \to Y$ is equivalent to $f: X \to Y$,*

ii) *$\tau_0: T_0(M) \to H^0(X, \mathcal{T})$ is bijective.*

The proof of each theorem is analogous to that of the corresponding theorem in [2], [3].

3. General case. Let $\{U_i\}$ be a fixed finite Stein covering of X. In the situation of section 1, if we do not assume that f is non-degenerate, we must replace $H^0(X, \mathcal{T})$ by

$$D_{X/Y}=\frac{\begin{array}{c}\{(\tau_i, \rho_{ij}): \tau_i \in \Gamma(U_i, f^*\Theta_Y), \rho_{ij} \in \Gamma(U_i \cap U_j, \Theta_X) \\ \tau_j-\tau_i=F\rho_{ij}, \rho_{jk}-\rho_{ik}+\rho_{ij}=0\}\end{array}}{\{(Fg_i, g_j-g_i): g_i \in \Gamma(U_i, \theta_X)\}}.$$

Then we can define a characteristic map

$$\tau: T_0(M) \to D_{X/Y}.$$

Theorem 1′. *In the situation of Theorem* 1, *we do not assume that f is non-degenerate. If $\tau: T_0(M) \to D_{X/Y}$ is surjective, then the family is complete at* 0.

Theorem 2′. *Let $f: X \to Y$ be a holomorphic map. If*

$$H^1(X, \mathcal{T})=0 \quad \text{and} \quad H^2(X, \Theta_{X/Y})=0$$

($\Theta_{X/Y}$ is the sheaf of germs of relative vector fields), then there exist a family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y and a point $0 \in M$ such that

i) *$\Phi_0: X_0 \to Y$ is equivalent to $f: X \to Y$,*

ii) *$\tau: T_0(M) \to D_{X/Y}$ is bijective.*

4. Costabilities.

Theorem 3. *Let $f: X \to Y$ be a holomorphic map. Suppose that*

i) *$f^*: H^1(Y, \Theta_Y) \to H^1(X, f^*\Theta_Y)$ is surjective,*

ii) *$f^*: H^2(Y, \Theta_Y) \to H^2(X, f^*\Theta_Y)$ is injective.*

Then for any family $p: \mathcal{X} \to M$ of deformations of $X=X_0 (0 \in M)$, there exist a family $q: \mathcal{Y} \to M$ of deformations of $Y=Y_0$ and a holomorphic map $\Phi: \mathcal{X} \to \mathcal{Y}$ over M such that Φ_0 coincides with f (restricting M on a neighborhood of 0 *if necessary).*

The relative version of Theorem 3 is

Theorem 4. *Let $f: X \to Y, g: Y \to Z$ be holomorphic maps, and let $h=g \circ f$. Assume that*

i) $f^*: H^0(Y, \mathcal{T}_{Y/Z}) \to H^0(X, f^*\mathcal{T}_{Y/Z})$ *is surjective,*

ii) $f^*: H^1(Y, \mathcal{T}_{Y/Z}) \to H^1(X, f^*\mathcal{T}_{Y/Z})$ *is injective,*

iii) $f^*: H^1(Y, \Theta_Y) \to H^1(X, f^*\Theta_Y)$ *is injective.*

Then for any commutative diagram

$$(*) \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\Upsilon} & \mathcal{Z} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{s} & S \end{array} \quad \text{with} \quad \begin{array}{l} X = X_0 (0 \in M), Z = Z_{0'} (0' \in S) \\ \Upsilon_0 = h \text{ and } s(0) = 0' \end{array}$$

there exists a family $q: \mathcal{Y} \to M$ such that the diagram $(*)$ is factored into

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ & {\scriptstyle p}\searrow & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ & & M & \longrightarrow & S \end{array}$$

(restricting M on a neighborhood of 0 in M if necessary).

5. Applications. I) Equi-blowing-down. The following theorem is an immediate consequence of Theorem 3.

Theorem 5. *Let $f: X \to Y$ be a monoidal transformation with a non-singular center D. Then for any family $p: \mathcal{X} \to M$ of deformations of $X = X_0 (0 \in M)$, there exist a family $q: \mathcal{Y} \to M$ of deformations of $Y = Y_0$, a holomorphic map $\Phi: \mathcal{X} \to \mathcal{Y}$ over M, a family $\mathcal{D} \to M$ of deformations of $D = D_0$ and an embedding $J: \mathcal{D} \to \mathcal{Y}$ over M such that*

i) $\Phi_0: X_0 \to Y_0$ *coincides with* $f: X \to Y$,

ii) $J_0: D_0 \to Y_0$ *coincides with* $D \subsetneq Y$,

iii) $\Phi_t: X_t \to Y_t$ *is the monoidal transformation with center* D_t *for* $t \in M$

(*restricting M on a neighborhood of* 0 *if necessary*).

II) Deformations of algebraic manifolds with ample canonical bundle.

We say that a compact complex manifold X is unobstructed if there exists a family $p: \mathcal{X} \to M$ of deformations of $X = X_0 (0 \in M)$ such that the infinitesimal deformation map

$$\rho: T_0(M) \to H^1(X, \Theta_X)$$

is surjective (cf. [1]). X is called obstructed if it is not unobstructed.

We give here an example of an obstructed X which has ample canonical bundle. By a result of Mumford [5], we can find a monoidal transformation $Y \to \boldsymbol{P}^3$ whose center is a non-singular space curve γ of degree 14 and of genus 24 such that Y is obstructed. Let X be a hypersurface in Y of sufficiently high order, then the canonical bundle of X is ample and X is obstructed; for if not, we can prove that Y is also unobstructed by virtue of Theorem 3, which is a contradiction.

Remark. The surface X constructed above is a non-singular model of a singular hypersurface X' in $\boldsymbol{P}^3$ of order ν which has γ as an m-fold curve $(\nu \gg 0, m \gg 0)$.

If we assume that X is a submanifold of an abelian variety and that the canonical bundle is ample, then we can prove that X is unobstructed by induction on $\dim X$, by virtue of Theorem 4.

References

[1] Kodaira, K., and Spencer, D. C.: On deformations of complex analytic structures. I, II. Ann. of Math., **67**, 328–466 (1958).

[2] ——: A theorem of completeness for complex analytic fibre spaces. Acta Math., **100**, 281–294 (1958).

[3] Kodaira, K., Nirenberg, L., and Spencer, D. C.: On the existence of deformations of complex analytic structures. Ann. of Math., **68**, 450–459 (1958).

[4] Kodaira, K.: A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. Ann. of Math., **75**, 146–162 (1962).

[5] Mumford, D.: Further pathologies in algebraic geometry. Amer. Jour. of Math., **84**, 642–648 (1962).