71. On the Integral of Cauchy-Stieltjes Type and I. I. Privalov's Fundamental Lemma. II

By Chuji TANAKA
Mathematical Institute, Waseda University, Tokyo

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3. **Fundamental Lemma 2.** In Fundamental Lemma 1, we have assumed that there exist one-sided derivatives: $F'_{\pm}(s_0)$. If we assume that L has a finite curvature at x_0, then without the existence of $F'_{\pm}(s_0)$ we can prove

Fundamental Lemma 2. *Suppose that $x(s)$ is twice continuously differentiable in the neighbourhood of s_0. For a fixed α, put*

$$z=x_0+\varepsilon e^{i\varphi_0}\cdot e^{i\alpha}(0<\alpha<\pi), \qquad z^*=x_0+\varepsilon^* e^{i\varphi_0}\cdot e^{-i\alpha^*}(0<\alpha^*<\pi),$$

where

(1) $x_0=x(s_0)$, $\varphi_0=\varphi(s_0)$,

(2) $\varepsilon^*\to 0$, $\alpha^*\to\alpha$ *as* $\varepsilon\to+0$ *in such a manner that* $\varepsilon^* e^{-i\alpha^*}=\varepsilon e^{-i\alpha}(1+0(\varepsilon))$.

Then, putting $\varepsilon e^{i\alpha}=x+iy$, following propositions hold;

(1) $$\lim_{\varepsilon\to+0}\{f(z)-f(z^*)\}=A \rightleftarrows \lim_{\varepsilon\to+0}\frac{1}{\pi}\cdot\int_{-h}^{+h}\frac{y}{(\sigma-x)^2+y^2}dF(s_0+\sigma)=A,$$

where A: a finite complex number, h: any fixed positive constant.

(2) *If $F(s)$ is continuous at $s=s_0$, then*

$$\lim_{\varepsilon\to+0}\left\{f(z)+f(z^*)-\frac{1}{\pi i}\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x(s)-x_0}\right\}=0 \rightleftarrows$$

$$\lim_{\varepsilon\to+0}\left\{\int_{-h}^{h}\frac{\sigma-x}{(\sigma-x)^2+y^2}dF(s_0+\sigma)-\int_{\varepsilon}^{h}\frac{1}{\sigma}d(F(s_0+\sigma)+F(s_0-\sigma)\right\}=0,$$

where h: any fixed positive constant.

(3) *If* $\alpha=\frac{\pi}{2}$, i.e. $x=0$, $y=\varepsilon$, *then next estimation holds:*

$$f(z)+f(z^*)-\frac{1}{\pi i}\cdot\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x(s)-x_0}$$
$$=0\left(\int_0^h\frac{\varepsilon}{\sigma^2+\varepsilon^2}\,|d(F(s_0+\sigma)+F(s_0-\sigma))|\right)+0\left(\int_{-h}^{h}|dF(s_0+\sigma)|\right)+o(1)$$

as $\varepsilon\to+0$, where h: any fixed positive constant.

From this lemma, we can derive some important boundary behaviours of the integral of Cauchy-Stieltjes type. We begin with

Corollary 2. *Assume that the conditions in Fundamental Lemma 2 are satisfied. Then following propositions hold;*

(1) *If there exists the finite symmetric derivative at s_0:*

$$\lim_{t\to+0}\frac{1}{2t}\{F(s_0+t)-F(s_0-t)\}=A,$$

then the radial limit exists:

$$\lim_{\varepsilon\to+0}(f(z)-f(z^*)=A,$$

where $z=x_0+i\varepsilon e^{i\varphi_0}$, $z^*=x_0-i\varepsilon^* e^{i\varphi_0}$.

(2) *If $F(s)$ is continuous and "smooth" at s_0*:

$$\lim_{t\to+0}\frac{1}{t}\{F(s_0+t)+F(s_0-t)-2F(s_0)\}=0,$$

then the radial limit exists:

$$f(z)+f(z^*)-\frac{1}{\pi i}\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x-x_0}\to 0$$

as $\varepsilon\to+0$ where $z=x_0+i\varepsilon e^{i\varphi_0}$, $z^*=x_0-i\varepsilon^* e^{i\varphi_0}$.

Corollary 3. *The necessary and sufficient condition for the existence of next finite chordal limit*:

$$\lim_{\varepsilon\to+0}\frac{1}{2\pi}\cdot\int_0^{2\pi}\frac{1-|z|^2}{|e^{is}-z|^2}dF(s)=A,$$

where $z=e^{is_0}+i\varepsilon\cdot e^{is_0}\cdot e^{i\alpha}(0<\alpha<\pi)$ is that we have

$$\lim_{\varepsilon\to+0}\frac{1}{\pi}\cdot\int_{-h}^{+h}\frac{y}{(\sigma-x)^2+y^2}dF(s_0+\sigma)=A,$$

where $\varepsilon e^{i\alpha}=x+iy$, and h: any positive fixed constant.

Now we introduce

Definition 1. *$F(s)$ is said to belong to $I_{a,b}$ at s_0 (for brevity $F(s)\in I_{a,b}$), if, putting $F(s)-(a+ib)(s-s_0)=U(s)+iV(s)$, $U(s)$ and $V(s)$ are increasing functions of s in the neighbourhood of s_0, where a and b are fixed finite real constants dependent upon s_0.*

Then we can prove the converse of Fatou-type theorem;

Theorem 2. *Under the same conditions as in Fundamental Lemma 2, assume further that $F(s)\in I_{a,b}$ at s_0. Then the converse of Fatou-type theorem holds, where A is a finite complex constant*;

(1) *If the radial limit exists*:

$$\lim_{\varepsilon\to+0}\{f(z)-f(z^*)\}=A,$$

where $z=x_0+i\varepsilon e^{i\varphi_0}$, $z^*=x_0-i\varepsilon^* e^{i\varphi_0}$, then $F(s)$ has the symmetric derivative at s_0:

$$\lim_{t\to+0}\frac{1}{2t}\{F(s_0+t)-F(s_0-t)\}=A.$$

(2) *If the angular limit exists*:

$$\lim_{\varepsilon\to+0}\{f(z)-f(z^*)\}=A,$$

where $z=x_0+\varepsilon e^{i\varphi_0}\cdot e^{i\alpha}$, $z^*=x_0+\varepsilon^* e^{i\varphi_0}\cdot e^{-i\alpha^*}$ $(0<\alpha<\pi,\ 0<\alpha^*<\pi)$ and $|\cos\alpha|\leqq q<1$ (q: a fixed positive constant), then $F(s)$ has the derivative at s_0: $F'(s_0)=A$.

Next theorem concerns with interesting boundary behaviour of Cauchy-Stieltjes integral.

Theorem 3. *Let $f(z)$ be the regular function defined by Cauchy-Stieltjes integral:*

$$f(z)=\frac{1}{2\pi i}\int_L \frac{e^{i\varphi}dF(s)}{x-z} \text{ for } z \text{ inside } L,\ f(z)\equiv 0 \text{ for } z \text{ outside } L.$$

If $x(s)$ is twice continuously differentiable in the neighbourhood of s_0, and $F(s)\in I_{a,b}$ at s_0, then following three propositions are equivalent, where A is a finite complex number;

(1) *$f(z)$ has the angular limit A at x_0.*

(2) *$F(s)$ has the derivative at s_0: $F'(s_0)=A$.*

(3) *The limit: $\lim_{\varepsilon\to+0}\int_\varepsilon^h \frac{1}{\sigma}d(F(s_0+\sigma)+F(s_0-\sigma))$ exists, and we have*

$$\frac{1}{\pi i}\int_L \frac{e^{i\varphi}dF(s)}{x-x_0}=A.$$

By virtue of Theorem 3, we can prove very remarkable results on H_1 class. We first introduce

Definition 2. *The complex-valued function $f(z)=u(z)+iv(z)$ is said to belong to $B_{a,b}$ at z_0 (for brevity $f(z)\in B_{a,b}$), if $u(z)>a$, $v(z)>b$ in the neighbourhood of z_0 contained in its definition-domain, where a and b are fixed finite real constants dependent upon z_0.*

Then next theorem holds;

Theorem 4. *Suppose that $f(z)\in H_1$ for $|z|<1$, and $f(z)\in B_{a,b}$ at e^{is_0}. Then following three propositions are equivalent, where A is a finite complex number;*

(1) *$f(z)$ has the angular limit A at e^{is_0}.*

(2) *Next limit exists: $\lim_{t\to 0}\frac{1}{t}\cdot\int_0^t f(e^{i(s_0+\tau)})d\tau=A$.*

(3) *The limit: $\lim_{\varepsilon\to+0}\int_\varepsilon^\pi \frac{1}{\tau}\{f(e^{i(s_0+\tau)})-f(e^{i(s_0-\tau)})\}d\tau$ exists, and we have*

$$\frac{1}{\pi i}\oint_{|x|=1}\frac{f(x)}{x-e^{is_0}}dx=A.$$

As an immediate consequence of Theorem 4, we have

Corollary 4. *Let $f(z)$ be regular and bounded for $|z|<1$. Then three conditions in Theorem 4 are equivalent.*

The equivalence of (1) and (2) in Corollary 4 is already known ([4] p. 119, [1] p. 612).

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