68. On Normal Approximate Spectrum. II

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1. Introduction. Suppose that a (bounded linear) operator T acts on a Hilbert space $\mathfrak{H}$. A complex number λ is an *approximate propervalue* of T if there exists a sequence $\{x_n\}$ of unit vectors such that

(*) $$\|(T-\lambda)x_n\| \to 0 \qquad (u \to \infty).$$

The set of all approximate propervalues of T is called the *approximate spectrum* $\pi(T)$ of T. According to Kasahara and Takai [8], an approximate propervalue λ of T is called *normal* if λ satisfies furthermore

(**) $$\|(T-\lambda)^* x_n\| \to 0 \qquad (n \to \infty).$$

The set $\pi_n(T)$ of all normal approximate propervalues of T is called the *normal approximate spectrum* of T. Several equivalent conditions which give the normal approximate spectra are discussed in [4] and [8].

In the present note, we shall concern with some additional properties of the normal approximate spectra of operators. In § 2, we shall give a theorem of Hildebrandt [7; Satz 2 (ii)] and observe its consequences. A theorem of Arveson [1; Theorem 3.1.2] follows at once. A theorem of Stampfli [9] is improved. In § 3, we shall show that a spectraloid is finite in the sense of Williams [10]. In § 4, we shall discuss a variant of a proposition of Bunce [3; Proposition 6].

2. Consequences of Hildebrandt's theorem. Hildebrandt [7] stated without proof the following theorem:

Theorem 1 (Hildebrandt). *If λ belongs to $\partial W(T)$ and $\pi(T)$, then $\lambda \in \pi_n(T)$, where $\partial W(T)$ is the frontier of the numerical range*

(1) $$W(T) = \{(Tx \mid x);\ \|x\| = 1\}.$$

Proof. We can assume that $\lambda = 0$ and $\mathrm{Re}\, T \geqq 0$ where

(2) $$\mathrm{Re}\, T = \frac{1}{2}(T + T^*).$$

Then we have

$$|(Tx_n \mid x_n)| \leqq \|Tx_n\| \to 0 \qquad (n \to \infty),$$

so that we have

$$(\mathrm{Re}\, Tx_n \mid x_n) \to 0 \qquad (n \to \infty).$$

Let A be the (positive) square-root of $\mathrm{Re}\, T$. Then we have

$$\|Ax_n\|^2 = (A^2 x_n \mid x_n) = (\mathrm{Re}\, Tx_n \mid x_n) \to 0 \qquad (n \to \infty).$$

Therefore we have

$$\|\mathrm{Re}\, Tx_n\| = \|A^2 x_n\| \to 0 \qquad (n \to \infty),$$

so that $\|T^*x_n\|$ converges to 0, by (2).

Remark. In the stage of the preparation of the present note, Prof. H. Choda kindly pointed out that a similar proof is presented by T. Saito in his lecture note at Tulane University (1970).

In the remainder of this section, we shall discuss some consequences of Theorem 1. At first, we shall show the following theorem due to [1; Theorem 3.1.2]:

Theorem 2 (Arveson). *If $|\lambda|=r(T)=w(T)$ and $\lambda\in\pi(T)$, where $r(T)$ is the spectral radius of T and $w(T)$ the numerical radius of T defined by*

$$(3)\qquad w(T)=\sup\{|\lambda|;\lambda\in W(T)\},$$

then there is a character ϕ on the unital C^*-algebra $\mathfrak{A}$ generated by T such that

$$(4)\qquad \phi(T)=\lambda.$$

Proof. By Theorem 1, the hypothesis implies that λ is a normal approximate propervalue of T. On the other hand, Kasahara and Takai [8] established extending a theorem of Bunce [2], for any $\lambda\in\pi_n(T)$ there is a character ϕ of $\mathfrak{A}$ satisfying (4), so that the theorem is proved.

Arveson [1; Corollary 3.1.3] shows that T is not normaloid (cf. [5]) if $\mathfrak{A}$ has no character. We shall improve Arveson's result into the following:

Theorem 3. *If $\mathfrak{A}$ has no character, then the generator T is not a spectraloid in the sense of Halmos* [5; p. 115], *that is,*

$$(5)\qquad r(T)<w(T).$$

Proof. If T is a spectraloid, i.e. $r(T)=w(T)$, then there is $\lambda\in\pi(T)$ satisfying $|\lambda|=r(T)=w(T)>0$. By Theorem 2, we have a non-trivial character ϕ of $\mathfrak{A}$ satisfying (4) and a contradiction.

Now, we shall introduce a class of operators due to [9]:

Definition 1 (Stampfli). An operator T acting on $\mathfrak{H}$ is called to belong the class $\boldsymbol{S}$ provided that $\pi_n(T)$ is not empty.

By a theorem of Kasahara-Takai [8] and our previous [4], we have at once the following

Corollary 1. *$T\in\boldsymbol{S}$ if and only if the unital C^*-algebra $\mathfrak{A}$ generated by T has a non-trivial character.*

Stampfli [9] proved that $\boldsymbol{S}$ is the uniform closure of the class $\boldsymbol{R}_1$ which is the set of all operators having one-dimensional reducing subspaces. He also proved that $\boldsymbol{S}$ contains all compact operators, all hyponormal operators and all Toeplitz operators. As an addition to Stampfli's results, we shall show the following fact:

Theorem 4. *All spectraloids belong to $\boldsymbol{S}$.*

Proof. Theorem 1 shows that $\pi_n(T)$ is not empty if T is a spectraloid; hence we have $T\in\boldsymbol{S}$.

3. Finite operators. Let us denote

(6) $$[A,B]=AB-BA$$

and introduce the following notion due to [10]:

Definition 2 (Williams). An operator T acting on a separable Hilbert space $\mathfrak{H}$ is *finite* if T satisfies one of the following conditions:

(7) $$0\in \bar{W}([T,A]) \qquad (A\in \mathfrak{B}(\mathfrak{H})),$$

(8) $$\inf_{A\in\mathfrak{B}(\mathfrak{H})} \|[T,A]-I\|=1,$$

(9) $$\rho(TA)=\rho(AT) \qquad (A\in\mathfrak{B}(\mathfrak{H}))$$

for a certain state ρ, where $\bar{W}$ is the closure of W and $\mathfrak{B}(\mathfrak{H})$ is the algebra of all operators acting on $\mathfrak{H}$. Let $\boldsymbol{F}$ be the set of all finite operators.

Using [5; Problem 185], Williams [10] pointed out that a normaloid is finite. We shall generalize this into the following form:

Theorem 5. *If T is a spectraloid, then T is finite.*

Proof. In the previous note [4], we have pointed out that $\lambda\in\pi_n(T)$ if and only if

(10) $$0\in \bar{W}((A-\lambda)^*(A-\lambda)+(A-\lambda)(A-\lambda)^*).$$

If T is a spectraloid, then we have a λ in $\partial W(T)\cap\pi(T)$, so that we have $\lambda\in\pi_n(T)$ and (10); hence there is a character ϕ on $\mathfrak{A}$ satisfying (4) by Kasahara-Takai's theorem [8]. Therefore we have $\phi((T-\lambda)^*(T-\lambda))=0$. Let us denote by the same ϕ an extension of ϕ on $\mathfrak{B}(\mathfrak{H})$. Then we have by the Schwarz inequality

$$|\phi(A(T-\lambda))|^2\leqq\phi(|T-\lambda|^2)\phi(AA^*)=0$$

for any $A\in\mathfrak{B}(\mathfrak{H})$. Hence we have

$$\phi(AT)=\lambda\phi(A)=\phi(A)\phi(T)=\phi(T)\phi(A)=\lambda\phi(A)=\phi(TA),$$

so that we can conclude that T is finite by (9).

Remark. We have given a direct proof of Theorem 5. However, we can give an another proof basing on a theorem of Williams [10; Theorem 6] which states that $\boldsymbol{S}\subset\boldsymbol{F}$. Since a spectraloid T belongs to $\boldsymbol{S}$ by Theorem 4, T is finite.

Here we note that *there is a finite operator which has no normal approximate propervalue.* Let

$$T=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Then T generates $\mathfrak{B}(\boldsymbol{C}^2)$, the algebra of all operators on the 2-dimensional space $\boldsymbol{C}^2$. Since $\mathfrak{B}(\boldsymbol{C}^2)$ has a trace, T is finite by (9). Whereas $\mathfrak{B}(\boldsymbol{C}^2)$ has no character since it is simple. Hence, T has no normal approximate propervalue by a theorem due to Kasahara and Takai [8].

Following after Arveson [1; § 3.2], let us introduce that an operator is *simple* if it generates a simple unital C^*-algebra. Then the above argument shows that *the normal approximate spectrum of a simple operator is empty.*

4. Polar decomposition. Bunce [3; Proposition 6] discussed the decomposition of an approximate propervalue with respect to the polar decomposition

(11) $$T=U|T|.$$

We shall give a generalization:

Theorem 6. *If* (11) *is the polar decomposition of* T *and* $\alpha e^{i\theta}\in\pi_n(T)$, *then there are* $\alpha\in\pi(|T|)=\pi_n(|T|)$ *and* $e^{i\theta}\in\pi_n(U)$ *unless* $\alpha=0$.

Proof. By a proposition of Bunce [2; Proposition 7], there is a state ϕ on $\mathfrak{B}(\mathfrak{H})$ such as $\phi(T)=\alpha e^{i\theta}$. Using the construction of Bunce [2] and Kasahara-Takai [8], we can assume that ϕ satisfies

$$\phi(BT)=\phi(TB)=\phi(T)\phi(B)$$

for any $B\in\mathfrak{B}(\mathfrak{H})$. Also we can show that ϕ satisfies moreover

(12) $$\phi(BA)=\phi(AB)=\phi(A)\phi(B)$$

for any $A\in\mathfrak{A}$, the unital C^*-algebra generated by T, so that ϕ is a character of $\mathfrak{A}$. Hence by (11) we have

$$\phi(T)=\phi(U)\phi(|T|) \quad\text{and}\quad \phi(|T|)=\phi(U)\phi(T),$$

so that we have

$$\phi(T)=\phi(U)\phi(U^*)\phi(T).$$

Hence we have $\phi(U)\phi(U^*)=1$.

On the other hand, we have $\phi(|T|)=\alpha$, so that $\phi(U)=e^{i\theta}$. Since the unital (abelian) C^*-algebra generated by $|T|$ is contained in $\mathfrak{A}$, so that $\alpha\in\pi_n(|T|)=\pi(|T|)$, by [4].

Since $U^*T=U^*U|T|=|T|$, we have $\phi(|T|)=\phi(U^*U)\phi(|T|)$, so that we have $\phi(U^*U)=1=\phi(U)\phi(U^*)$. Therefore, we have

$$\phi((U-\phi(U))^*(U-\phi(U)))=0.$$

Hence $e^{i\theta}=\phi(U)\in\pi(U)$, cf. [4], [8]. On the other hand, $\|U\|=1$, so that $\phi(U)\in\pi_n(U)$, by [8; Theorem 5].

Remark. Theorem 6 is easily proved by assuming that T is invertible. Since $|T|$ is in $\mathfrak{A}$ and invertible in $\mathfrak{A}$, $U=T|T|^{-1}\in\mathfrak{A}$. Hence there is a character ϕ on $\mathfrak{A}$ such that

(13) $$\alpha e^{i\theta}=\phi(T)=\phi(U|T|)=\phi(U)\phi(|T|).$$

Since $\phi(|T|)=\alpha$, we have $\phi(U)=e^{i\theta}$, and we conclude the theorem.

5. Pseudoradical. In the previous note [4], we proved a theorem of Coburn based on the reciprocity of the characters and the normal approximate spectrum of an operator. In this section, we shall discuss the method of [4] in a general setting.

Let $\mathfrak{B}$ be a unital C^*-algebra acting on $\mathfrak{H}$. Following after Arveson [1; §3.3], we shall introduce the closed two-sided ideal $\mathfrak{R}$ which is generated by $\{[A,B]; A,B\in\mathfrak{A}\}$. We shall call $\mathfrak{R}$ the *pseudoradical* of $\mathfrak{B}$. Then $\mathfrak{B}/\mathfrak{R}$ *is an abelian* C^*-*algebra.* By the Gelfand representation, $\mathfrak{B}/\mathfrak{R}$ is isometrically isomorphic with the algebra $C(X)$ of all continuous functions defined on a compact Hausdorff space X which is homeomorphic with the character space of $\mathfrak{B}/\mathfrak{R}$. Hence the name of $\mathfrak{R}$ may be justified by the fact that $\mathfrak{R}$ corresponds to the radical in the theory of commutative Banach algebras.

Examining our previous proof of Coburn's theorem, we shall show the following theorem:

Theorem 7. *If $\mathfrak{A}$ is the unital C^*-algebra generated by T, then $\mathfrak{A}/\mathfrak{R}$ is isometrically isomorphic to $C(\pi_n(T))$.*

Proof. Our proof is essentially same with a part of proof of Coburn's theorem in [4; § 4], so that we shall give a brief sketch. At first, we notice that the kernel of a character ϕ contains $\mathfrak{R}$ since $\phi(AB) = \phi(A)\phi(B) = \phi(BA)$, so that we have $\mathfrak{R} \subset \mathfrak{N}$ where $\mathfrak{N}$ is the intersection of the kernels of all characters of $\mathfrak{A}$. If $\mathfrak{N}$ properly contains $\mathfrak{R}$, then there is a character ϕ' on $\mathfrak{A}/\mathfrak{R}$ which does not vanish on $\mathfrak{N}/\mathfrak{R}$, so that the natural extension ϕ of ϕ' on $\mathfrak{A}$ is a character of $\mathfrak{A}$ and the kernel of ϕ excludes $\mathfrak{N}$, and we have a contradiction; hence $\mathfrak{N} = \mathfrak{R}$. By the reciprocity among the characters of $\mathfrak{A}$ and the normal approximate spectrum of T, we have that $\mathfrak{A}/\mathfrak{R}$ is isomorphic to $C(\pi_n(T))$.

Remark. In general, the pseudoradical $\mathfrak{R}$ does not coincide with the algebra $\mathfrak{C}(\mathfrak{H})$ of all compact operators on $\mathfrak{H}$. However, we can prove the following fact due to Arveson [1; Corollary 3.3.7]: *If T is irreducible and almost normal* (i.e. $T^*T - TT^* \in \mathfrak{C}(\mathfrak{H})$), *then the pseudoradical $\mathfrak{R}$ is equal to $\mathfrak{C}(\mathfrak{H})$.* Using this and Theorem 7, we can deduce Coburn's theorem.

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