64. On $u_t = u_{xx} - F(u_x)$ and the Differentiability of the Nonlinear Semi-Group Associated with it

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1. **Introduction.** Suggested by a problem which has to do with the *burning of gas in a rocket* (see Forsythe and Wasow [4], p. 141), we consider in the present paper the following problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - F\left(\dfrac{\partial u}{\partial x}\right) & \text{in } (-\pi, \pi)\times(0,\infty), \\ u(-\pi, t) = u(\pi, t) & \text{in } (0,\infty), \\ u(x,0) = u_0(x) & \text{in } (-\pi, \pi), \end{cases}$$

where F is a *continuous* function on R^1 such that $F(0)=0$. We shall prove the existence and the uniqueness theorem[1)] for solutions of (1.1) by studying the *differentiability of the nonlinear contraction semi-group* on $C_{2\pi}[-\pi,\pi]\subset L^\infty(-\pi,\pi)$, associated with (1.1), which is generated in the sense of Theorem I of Crandall and Liggett [3]; here $C_{2\pi}[-\pi,\pi]$ is the Banach space of all real-valued continuous functions f on $[-\pi,\pi]$ satisfying $f(-\pi)=f(\pi)$, endowed with the norm $\|\cdot\|_\infty$ induced by $L^\infty(-\pi,\pi)$.

2. Our result reads:

Theorem. *Assume that*

$$(2.1)\qquad u_0, \frac{du_0}{dx} \in C_{2\pi}[-\pi,\pi] \quad \textit{and} \quad \frac{d^2u_0}{dx^2} \in L^\infty(-\pi,\pi).$$

Then the equation (1.1) *has a unique solution* $u=u(x,t)$ *such that*

$$(2.2)\qquad u, \frac{\partial u}{\partial x} \in C([0,\infty); C_{2\pi}[-\pi,\pi]), \qquad \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \in L^\infty((-\pi,\pi)\times(0,\infty));$$

where $\partial u/\partial x, \partial^2 u/\partial x^2, \partial u/\partial t$ *denote the distribution derivatives of* $u\in\mathcal{D}'((-\pi,\pi)\times(0,\infty))$.

3. **The uniqueness.** We set, for $1<p\leqslant\infty$,

$$\tau_p(f,g) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(\|f+\varepsilon g\|_p - \|f\|_p),\ f, g\in L^p(-\pi,\pi).\text{[2)]}$$

Then, by Sato [14], § 6, we have

$$\tau_\infty(f,g) = \max_{x\in\{x;|f(x)|=\|f\|_\infty\}} (\operatorname{sgn} f(x))g(x), \quad f,g\in C_{2\pi}[-\pi,\pi], \quad f\neq 0,$$

and, for $1<p<\infty$,

1) Another approach to a similar problem is seen, for example, in Kruzhkov [12].

2) $\|\cdot\|_p = \|\cdot\|_{L^p(-\pi,\pi)}$.

$$\tau_p(f,g)=\int_{-\pi}^{\pi}(\operatorname{sgn} f(x))|f(x)|^{p-1}g(x)dx/\|f\|_p^{p-1}, \quad f,g\in L^p(-\pi,\pi), \quad f\neq 0.$$

We shall need the following:

Lemma 1. *Suppose that $f, g\in C_{2\pi}[-\pi,\pi]$. Then*

$$\varlimsup_{p\to\infty}\tau_p(f,g)\leqslant\tau_\infty(f,g). \tag{3.1}$$

Proof. Without loss of generality, we can assume that $\|f\|_\infty=1$. We have

$$\begin{aligned}
&\int_{-\pi}^{\pi}(\operatorname{sgn} f(x))|f(x)|^{p-1}g(x)dx\\
&\quad=\int_{\{x;|f(x)|=1\}}(\operatorname{sgn} f(x))g(x)dx+\int_{\{x;|f(x)|<1\}}(\operatorname{sgn} f(x))|f(x)|^{p-1}g(x)dx\\
&\quad\leqslant\max_{x\in\{x;|f(x)|=1\}}(\operatorname{sgn} f(x))g(x)\cdot\mu(\{x;|f(x)|=1\})+\|g\|_\infty\cdot\int_{\{x;|f(x)|<1\}}|f(x)|^{p-1}dx
\end{aligned}$$

and

$$\|f\|_p^{p-1}=[\mu(\{x;|f(x)|=1\})+\int_{\{x;|f(x)|<1\}}|f(x)|^p dx]^{(p-1)/p},$$

where μ is the Lebesgue measure on $(-\pi,\pi)$. Hence, by the bounded convergence theorem, we have (3.1). Q.E.D.

Proof of Theorem (uniqueness). Let $u^{(1)}$ and $u^{(2)}$ be solutions of (1.1) satisfying (2.2) with $u=u^{(1)}, u^{(2)}$. Note that, for $1<p<\infty$,

$$\tau_p\left(u^{(1)}(t)-u^{(2)}(t), \frac{\partial^2}{\partial x^2}u^{(1)}(t)-\frac{\partial^2}{\partial x^2}u^{(2)}(t)\right)\leqslant 0 \qquad \text{for a.a. } t>0$$

(cf. Theorem 1 of Hasegawa [5]). Since $u^{(1)}(t)-u^{(2)}(t)$ is strongly differentiable in $L^p(-\pi,\pi)$ $(1<p<\infty)$ for a.a. $t>0$, we have

$$\begin{aligned}
&\frac{d}{dt}\|u^{(1)}(t)-u^{(2)}(t)\|_p\\
&\quad=\tau_p\left(u^{(1)}(t)-u^{(2)}(t), \frac{\partial u^{(1)}}{\partial t}(t)-\frac{\partial u^{(2)}}{\partial t}(t)\right) \qquad \text{(cf. Lemma 1.3 of Kato [6])}\\
&\quad=\tau_p\left(u^{(1)}(t)-u^{(2)}(t), \frac{\partial^2}{\partial x^2}u^{(1)}(t)-\frac{\partial^2}{\partial x^2}u^{(2)}(t)\right)\\
&\qquad+\tau_p\left(u^{(1)}(t)-u^{(2)}(t), -F\left(\frac{\partial}{\partial x}u^{(1)}(t)\right)+F\left(\frac{\partial}{\partial x}u^{(2)}(t)\right)\right)\\
&\quad\leqslant\tau_p\left(u^{(1)}(t)-u^{(2)}(t), -F\left(\frac{\partial}{\partial x}u^{(1)}(t)\right)+F\left(\frac{\partial}{\partial x}u^{(2)}(t)\right)\right)
\end{aligned}$$

for a.a. $t>0$. Hence one obtains, for each $t\geqslant 0$,

$$\begin{aligned}
&\|u^{(1)}(t)-u^{(2)}(t)\|_p\\
&\qquad\leqslant\int_0^t\tau_p\left(u^{(1)}(s)-u^{(2)}(s), -F\left(\frac{\partial}{\partial x}u^{(1)}(s)\right)+F\left(\frac{\partial}{\partial x}u^{(2)}(s)\right)\right)ds
\end{aligned}$$

and, letting p tend to infinity, by the Lebesgue-Fatou lemma and Lemma 1

$$\|u^{(1)}(t)-u^{(2)}(t)\|_\infty$$

$$\leqslant\int_0^t \tau_\infty\left(u^{(1)}(s)-u^{(2)}(s),\ -F\left(\frac{\partial}{\partial x}u^{(1)}(s)\right)+F\left(\frac{\partial}{\partial x}u^{(2)}(s)\right)\right)ds$$

$=0,\ t\geqslant 0$ (remember the form of τ_∞).

Consequently $u^{(1)}=u^{(2)}$ in $[-\pi,\pi]\times[0,\infty)$. Q.E.D.

4. Existence. We define an operator A in $C_{2\pi}[-\pi,\pi]$:

$$D(A)=\left\{u;\ u,\frac{du}{dx},\frac{d^2u}{dx^2}\in C_{2\pi}[-\pi,\pi]\right\},\qquad Au=-\frac{d^2u}{dx^2}+F\left(\frac{du}{dx}\right).$$

It is easily checked that A is *accretive* in $C_{2\pi}[-\pi,\pi]$ in the sense of Kato [7]:

$$\|u-v+\lambda(Au-Av)\|_\infty\geqslant\|u-v\|_\infty\qquad\text{for each }\lambda>0,\ u,v\in D(A)$$

(notice that $\tau_\infty(u-v,Au-Av)\geqslant 0$ for $u,v\in D(A)$). Moreover we have

Proposition. *The operator A defined above is m-accretive in $C_{2\pi}[-\pi,\pi]$. I.e. A is accretive and satisfies*

(4.1) $\qquad R(I+\lambda A)=C_{2\pi}[-\pi,\pi]\qquad$ *for each* $\lambda>0$.

The proof relies upon

Lemma 2. *Suppose that*

$$u,\ du/dx\in C_{2\pi}[-\pi,\pi]\quad\text{and}\quad d^2u/dx^2\in L^\infty(-\pi,\pi).$$

Set

$$h=u-d^2u/dx^2+F(du/dx).$$

Then we have

$$\|du/dx\|_\infty\leqslant 2\pi\|h\|_\infty.$$

Proof. It is easy to see that

$$\|du/dx\|_\infty\leqslant\sqrt{2\pi}\,\|d^2u/dx^2\|_2.$$

On the other hand, we have

$$\left\|\frac{du}{dx}\right\|_2^2+\left\|\frac{d^2u}{dx^2}\right\|_2^2$$
$$=-\int_{-\pi}^{\pi}\left(u-\frac{d^2u}{dx^2}+F\left(\frac{du}{dx}\right)\right)\frac{d^2u}{dx^2}dx$$
$$=-\int_{-\pi}^{\pi}h\frac{d^2u}{dx^2}dx\leqslant\frac{1}{2}\|h\|_2^2+\frac{1}{2}\left\|\frac{d^2u}{dx^2}\right\|_2^2.$$

Thus

$$\|d^2u/dx^2\|_2^2\leqslant\|h\|_2^2\leqslant 2\pi\|h\|_\infty^2.$$ Q.E.D.

Let us introduce an operator Λ in $C_{2\pi}[-\pi,\pi]$:

$$D(\Lambda)=\left\{u;\ u,\frac{du}{dx}\in C_{2\pi}[-\pi,\pi]\right\},\qquad \Lambda u=\frac{du}{dx}.$$

It is well known that Λ is the infinitesimal generator of a group of class (C_0) on $C_{2\pi}[-\pi,\pi]$.

Proof of Proposition. *The first step.*[3] We shall prove (4.1) under the additional assumption that *F is Lipshitz continuous*:

$$\sup_{-\infty<r<s<\infty}\frac{|F(r)-F(s)|}{|r-s|}=L<\infty.$$

3) A similar technique is seen, for instance, in Yosida [16].

Set $0<\lambda_0<1/4L^2$. Since

$$u+\lambda_0 Au=v+\lambda_0 F(\Lambda(I-\lambda_0\Lambda^2)^{-1}v),\ u-\lambda_0\Lambda^2 u=v,$$

for each $u\in D(A)$ and

$$\|\lambda_0 F(\Lambda(I-\lambda_0\Lambda^2)^{-1}v_1)-\lambda_0 F(\Lambda(I-\lambda_0\Lambda^2)^{-1}v_2)\|_\infty \leqslant 2L\sqrt{\lambda_0}\,\|v_1-v_2\|_\infty \quad \text{for} \quad v_1, v_2\in C_{2\pi}[-\pi,\pi],$$

we have

$$R(I+\lambda_0 A)=C_{2\pi}[-\pi,\pi],$$

from which follows (4.1) (see Lemma 2.1 of Kato [6]).

The second step. We shall prove (4.1) *for general F*. We only have to prove (4.1) for $\lambda=1$ (Lemma 2.1 of Kato [6]). Let $\{F_n\}_{n\geqslant 1}$ be a sequence of Lipshitz continuous functions F_n on R^1 such that $F_n(0)=0$, which converges to F uniformly on any bounded interval of R^1. By the first step, for an arbitrarily fixed $h\in C_{2\pi}[-\pi,\pi]$, there exists $\{u_n\}_{n\geqslant 1}\subset D(\Lambda^2)$ satisfying

$$u_n-\Lambda^2 u_n+F_n(\Lambda u_n)=h, \qquad n\geqslant 1.$$

Since $A_n v=-\Lambda^2 v+F_n(\Lambda v)$ with $D(A_n)=D(\Lambda^2)$, is accretive in $C_{2\pi}[-\pi,\pi]$, we have

$$\|u_n\|_\infty\leqslant\|h\|_\infty, \qquad n\geqslant 1.$$

From Lemma 2 with $F=F_n$ follows the estimate:

$$\|\Lambda u_n\|_\infty\leqslant 2\pi\|h\|_\infty, \qquad n\geqslant 1.$$

Thus we have

$$\sup_{n\geqslant 1}\|\Lambda^2 u_n\|_\infty<\infty.$$

Accordingly there exists $u\in D(\Lambda)$ such that, for some subsequence $\{n'\}$ of $\{n\}$,

$$\operatorname*{s-lim}_{n'\to\infty} u_{n'}=u, \qquad \operatorname*{s-lim}_{n'\to\infty}\Lambda u_{n'}=\Lambda u$$

exist in $C_{2\pi}[-\pi,\pi]$. Moreover

$$\operatorname*{s-lim}_{n'\to\infty}\Lambda^2 u_{n'}=u+F(\Lambda u)-h \quad \text{in} \quad C_{2\pi}[-\pi,\pi].$$

Hence

$$u\in D(\Lambda^2) \quad \text{and} \quad \Lambda^2 u=u+F(\Lambda u)-h. \qquad \text{Q.E.D.}$$

We define an extension $\tilde{A}$ of A in $L^\infty(-\pi,\pi)$:

$$D(\tilde{A})=\left\{u;\ u,\frac{du}{dx}\in C_{2\pi}[-\pi,\pi] \ \text{and}\ \frac{d^2u}{dx^2}\in L^\infty(-\pi,\pi)\right\},$$

$$\tilde{A}u=-\frac{d^2u}{dx^2}+F\left(\frac{du}{dx}\right).$$

$\tilde{A}$ is accretive in $L^\infty(-\pi,\pi)$ and, by Proposition, satisfies

$$R(I+\lambda\tilde{A})\supset R(I+\lambda A)=C_{2\pi}[-\pi,\pi]=\overline{D(\tilde{A})},\ \lambda>0.$$

Thus $-\tilde{A}$ generates a nonlinear contraction semi-group $\{\exp(-t\tilde{A})\}_{t\geqslant 0}$ on $C_{2\pi}[-\pi,\pi]$ in the sense of Theorem I of Crandall and Liggett [3]: for $h\in C_{2\pi}[-\pi,\pi]$,

$$\exp(-t\tilde{A})\cdot h=\operatorname{s-lim}_{\lambda\downarrow 0}(I+\lambda\tilde{A})^{-[t/\lambda]}h \quad \text{in} \quad C_{2\pi}[-\pi,\pi],\ t\geqslant 0.^{4)}$$

We shall show that $u\in C([0,\infty);C_{2\pi}[-\pi,\pi])$ given by

$$u(t)=\exp(-t\tilde{A})\cdot u_0,\ t\geqslant 0,$$

is the solution mentioned in Theorem.

Proof of Theorem (existence). We define an extension $\tilde{A}$ of A in $L^\infty(-\pi,\pi)$:

$$D(\tilde{A})=\{u;\ u\in C_{2\pi}[-\pi,\pi] \quad \text{and} \quad du/dx\in L^\infty(-\pi,\pi)\}, \qquad \tilde{A}u=\frac{du}{dx}.$$

One knows the estimates:

$$\|(I+\lambda\tilde{A})^{-[t/\lambda]}u_0\|_\infty\leqslant\|u_0\|_\infty, \qquad \|\tilde{A}(I+\lambda\tilde{A})^{-[t/\lambda]}u_0\|_\infty\leqslant\|\tilde{A}u_0\|_\infty$$

for each $\lambda>0$ and $t\geqslant 0$; hence by Lemma 2

$$\|A(I+\lambda\tilde{A})^{-[t/\lambda]}u_0\|_\infty\leqslant 2\pi(\|u_0\|_\infty+\|\tilde{A}u_0\|_\infty) \qquad (\equiv\kappa)$$

and

$$\|\tilde{A}A(I+\lambda\tilde{A})^{-[t/\lambda]}u_0\|_\infty\leqslant\|\tilde{A}u_0\|_\infty+\max_{|r|\leqslant\kappa}F(r).$$

Consequently $u(t)\in D(\tilde{A})$,

$$\operatorname{s-lim}_{\lambda\downarrow 0}A(I+\lambda\tilde{A})^{-[t/\lambda]}u_0=Au(t) \quad \text{in} \quad C_{2\pi}[-\pi,\pi]$$

and

$$\operatorname{w^*-lim}_{\lambda\downarrow 0}\tilde{A}A(I+\lambda\tilde{A})^{-[t/\lambda]}u_0=\tilde{A}Au(t) \quad \text{in} \quad L^\infty(-\pi,\pi)$$

for each $t\geqslant 0$. Now letting λ tend to 0 in the estimate due to Ôharu (see (8) in [13]):

$$(I+\lambda\tilde{A})^{-[t/\lambda]}u_0-u_0=-\int_0^t\tilde{A}(I+\lambda\tilde{A})^{-[s/\lambda]}u_0ds+0(\lambda) \quad \text{strongly in } L^\infty(-\pi,\pi),$$

we have, by the Lebesgue's theorem,

$$u(t)-u_0=-\operatorname{w^*-}\int_0^t\tilde{A}u(s)ds \quad \text{in} \quad L^\infty(-\pi,\pi),\ t\geqslant 0,$$

i.e., by the weak* continuity of $t\mapsto\tilde{A}u(t)$ in $L^\infty(-\pi,\pi)$,

$$\operatorname{w^*-}\frac{d}{dt}u(t)=-\tilde{A}u(t) \quad \text{in} \quad L^\infty(-\pi,\pi),\ t\geqslant 0. \qquad \text{Q.E.D.}$$

5. Remarks. $D(\tilde{A})$ coincides with what Crandall [2] calls the "*generalized domain*" $\hat{D}(A)$ of A for our example.

$-\tilde{A}$ is "0-*dispersive*(*s*)" in $L^\infty(-\pi,\pi)$ in the sense of Konishi [9]. Hence $\{\exp(-t\tilde{A})\}_{t\geqslant 0}$ is an "*order-preserving semi-group*" (of type 0) on $C_{2\pi}[-\pi,\pi]$. See also Sato [15].

6. Comment. It is known (, for example, in Kōmura [8], Kato [7], Crandall and Liggett [3]) that if $\mathcal{A}$ is an m-accretive (multi-valued) operator in a reflexive Banach space, the semi-group $\{\exp(-t\mathcal{A})\}_{t\geqslant 0}$ generated by $-\mathcal{A}$ gives the unique strong solution to the problem:

4) $\{\exp(-t\tilde{A})\}_{t\geqslant 0}$ coincides with $\{\exp(-tA)\}_{t\geqslant 0}$, where, by definition, $\exp(-tA)\cdot h=\operatorname{s-lim}_{\lambda\downarrow 0}(I+\lambda A)^{-[t/\lambda]}h$ in $C_{2\pi}[-\pi,\pi]$, $t\geqslant 0$, $h\in C_{2\pi}[-\pi,\pi]$.

$$\begin{cases} \frac{du}{dt}(t)+\mathcal{A}u(t)\ni 0 & \text{a.a. } t\in(0,\infty), \\ u(0)=u_0 & (\in D(\mathcal{A})) ; \end{cases}$$

(set $u(t)=\exp(-t\mathcal{A})\cdot u_0$). But it is not the case in non-reflexive Banach spaces and the complete general theory on the differentiability of $\{\exp(-t\mathcal{A})\}_{t\geqslant 0}$ has not been established yet. Thus it seems interesting for us to study it case by case as in the present paper; see also Crandall [1] and Konishi [10, 11], where one will find the study in L^1 spaces.

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References

[1] M. G. Crandall: The semigroup approach to first order quasilinear equations in several space variables. Israel J. Math. (to appear).
[2] ——: A generalized domain for semigroup generators. MRC Technical Summary Report #1189, Madison, Wisconsin.
[3] M. G. Crandall and T. M. Liggett: Generation of semi-groups of nonlinear transformations on general Banach spaces. Amer. J. Math., **93**, 265–298 (1971).
[4] G. E. Forsythe and W. R. Wasow: Finite-Difference Methods for Partial Differential Equations. John Wiley and Sons, Inc., New York-London (1960).
[5] M. Hasegawa: On contraction semi-groups and (*di*)-operators. J. Math. Soc. Japan, **18**, 290–302 (1966).
[6] T. Kato: Nonlinear semigroups and evolution equations. J. Math. Soc. Japan, **19**, 508–520 (1967).
[7] ——: Accretive operators and nonlinear evolution equations in Banach spaces. Proc. Symp. in Pure Math., **18** (Part 1), 138–161 (1970).
[8] Y. Kōmura: Nonlinear semi-groups in Hilbert space. J. Math. Soc. Japan, **19**, 493–507 (1967).
[9] Y. Konishi: Nonlinear semi-groups in Banach lattices. Proc. Japan Acad., **47**, 24–28 (1971).
[10] ——: Some examples of nonlinear semi-groups in Banach lattices. J. Fac. Sci. Univ. Tokyo Sect. IA, **18**, 537–543 (1972).
[11] ——: Une méthode de résolution d'une équation d'évolution non linéaire dédénérée. J. Fac. Sci. Univ. Tokyo Sect. IA, **19** (to appear).
[12] S. N. Kruzhkov: Generalized solutions of nonlinear equations of first order with several variables. I (in Russian). Mat. Sbornik, **70**, 394–415 (1966).
[13] S. Ôharu: A note on the generation of nonlinear semigroups in a locally convex space. Proc. Japan Acad., **43**, 847–851 (1967).
[14] K. Sato: On the generators of non-negative contraction semi-groups in Banach lattices. J. Math. Soc. Japan, **20**, 423–436 (1968).
[15] ——: A note on nonlinear dispersive operators. J. Fac. Sci. Univ. Tokyo Sect. IA, **18**, 465–473 (1972).
[16] K. Yosida: On holomorphic Markov processes. Proc. Japan Acad., **42**, 313–317 (1966).