102. A Note on Infinitesimal Generators and Potential Operators of Contraction Semigroups

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1. Introduction. Hirsch [3] proves that an operator V in a Banach space is the "cogenerator" $\lim_{\lambda\to 0} J_\lambda$ of a pseudo-resolvent J_λ satisfying $\lim_{\lambda\to 0}\lambda J_\lambda=0$ if and only if $-V$ is the "generator" $\lim_{\lambda\to\infty}\lambda(\lambda J'_\lambda-1)$ of a pseudo-resolvent J'_λ satisfying $\lim_{\lambda\to\infty}\lambda J'_\lambda-1=0$. He notices a dual relation between J_λ and J'_λ. For semigroups, such a duality is not obtained between infinitesimal generators (i.g.) and potential operators (p.o.). However, the situation is rather simple in the case of contraction semigroups in Hilbert spaces, which is implicit in Hirsch [2]. In this note we give the result more explicitly, and also give a connection with Phillips' characterization of i.g. Further we consider contraction semigroups in Banach spaces.

2. Hilbert space. Let $\mathfrak{X}$ be a Hilbert space (real or complex). We mean by a contraction semigroup a strongly continuous semigroup of linear contraction operators on $\mathfrak{X}$. A contraction semigroup T_t with resolvent $J_\lambda(\lambda>0)$ and i.g. A is said to admit a p.o. if the set of f such that $J_\lambda f$ strongly converges as $\lambda\to 0$ is dense in $\mathfrak{X}$. If this is the case, the operator V defined by the limit is called the p.o. and satisfies $V=-A^{-1}$ (Yosida [11]). An operator A is called dissipative if $\mathrm{Re}\,(f, Af)\leqq 0$ for all $f\in\mathfrak{D}(A)$, and maximal dissipative if in addition no proper extension of it is dissipative. The Cayley transform C of A is defined by $C=(1+A)(1-A)^{-1}$ (Phillips [5]). $\mathfrak{D}, \mathfrak{R}$, and $\mathfrak{N}$ denote domain, range, and null space of an operator, respectively.

Theorem 1. *Let A be a linear operator in $\mathfrak{X}$. Then the following six conditions are equivalent:*

(i) *A is the i.g. of a contraction semigroup admitting a p.o.*

(ii) *$-A$ is the p.o. of a contraction semigroup.*

(iii) *A is maximal dissipative with $\mathfrak{D}(A)$ and $\mathfrak{R}(A)$ both dense.*

(iv) *A is dissipative, $\mathfrak{R}(1-A)=\mathfrak{X}$, and $\mathfrak{D}(A)$ and $\mathfrak{R}(A)$ are both dense.*

(v) *$(1-A)^{-1}$ is defined on $\mathfrak{X}$ and the Cayley transform C of A is a contraction operator with $\mathfrak{N}(C+1)=\mathfrak{N}(C-1)=0$.*

(vi) *There is a linear contraction operator C with $\mathfrak{D}(C)=\mathfrak{X}$ and $\mathfrak{N}(C+1)=\mathfrak{N}(C-1)=0$ such that $A=(C-1)(C+1)^{-1}$.*

Suppose that the above conditions are met, and let $T_t^{(1)}$ and $T_t^{(2)}$ be *the semigroups in* (i) *and* (ii), *respectively. Let* $A^{(i)}, V^{(i)}$, *and* $J_\lambda^{(i)}$ *be the i.g., p.o., and resolvent of* $T_t^{(i)}$ *and let* $C^{(i)}$ *be the Cayley transform of* $A^{(i)}(i=1,2)$. *Then, we have*

$$C^{(i)}=(V^{(i)}-1)(V^{(i)}+1)^{-1}. \tag{1}$$

We have for $i \neq j$

$$V^{(i)}=-A^{(j)}, \tag{2}$$

$$J_\lambda^{(i)}=\frac{1}{\lambda}\left(1-\frac{1}{\lambda}J_{1/\lambda}^{(j)}\right), \tag{3}$$

$$C^{(i)}=-C^{(j)}. \tag{4}$$

Remark. The property $\mathfrak{N}(C+1)=0$ in Conditions (v) and (vi) can be replaced by denseness of $\mathfrak{R}(C+1)$, since $\mathfrak{N}(C+1)=\mathfrak{N}(C^*+1)$ by [9] p. 8. Similarly, we can replace $\mathfrak{N}(C-1)=0$ by denseness of $\mathfrak{R}(C-1)$.

Proof. It is enough to combine together the results of Yosida [10] [11], Phillips [5], and Hirsch [2] [3]. It is found by Phillips [5] that the following five conditions are equivalent: (i)′ A is the i.g. of a contraction semigroup; (iii)′ A is maximal dissipative with dense domain; (iv)′ A is dissipative with dense domain and $\mathfrak{R}(1-A)=\mathfrak{X}$; (v)′ $(1-A)^{-1}$ is defined on $\mathfrak{X}$ and the Cayley transform C of A is a contraction with $\mathfrak{N}(C+1)=0$; (vi)′ there is a linear contraction operator C with $\mathfrak{D}(C)=\mathfrak{X}$ and $\mathfrak{N}(C+1)=0$ such that $A=(C-1)(C+1)^{-1}$. Keeping this equivalence in mind and recalling that a contraction semigroup admits a p.o. if and only if the i.g. has dense range (Yosida [11]), we see the equivalence of (i), (iii), (iv), (v) and (vi). It remains to prove the equivalence of (i) and (ii). Suppose that (i) holds for A, and let J_λ and V be its resolvent and p.o. We have $\lambda J_\lambda f \to 0$ as $\lambda \to 0$ for every f. We see that $(\lambda+V)^{-1}$ is defined on X since $\lambda+V=(1-\lambda A)V$. Further we have $(\lambda+V)^{-1}=(1/\lambda)(1-(1/\lambda)J_{1/\lambda})$. Hence $\lambda(\lambda+V)^{-1}f \to f$ as $\lambda \to \infty$ for every f. Now, since $\mathfrak{X}$ is Hilbert, we can prove $\|\mu J_\mu -1\| \leqq 1$ for all $\mu>0$. In fact, the contraction property of the Cayley transform of $\mu^{-1}A$ means a stronger result $\|\mu J_\mu-(1/2)\| \leqq 1/2$. Hence, by the Hille-Yosida theorem, $-V$ is the i.g. of a contraction semigroup, that is, (ii) holds. This argument also shows that (ii) implies (i). The relations (2) and (3) are now obvious. Since $(1-A)^{-1}=V(V+1)^{-1}$, we get (1) and then (4) follows from (1) and (2). The proof is complete.

Remark. Let $\mathfrak{X}$ be the real L^2 space on R^1. Using the notations of Theorem 1, we can prove that if $T_t^{(1)}$ is nonnegative, $T_t^{(2)}$ is not necessarily nonnegative. This is the case if $A^{(1)}$ is the closure of $B=d/dx$ or d^2/dx^2 with $\mathfrak{D}(B)$ being the C^∞ functions with compact supports, since $((Bf)^+, f) \leqq 0$ does not hold for $f \geqq 0$, $f \not\equiv 0$. Note that $((A^{(1)}f)^+, f)=(g^+, A^{(2)}g)$ for $g=A^{(1)}f$, and see [6] for this criterion.

2. Banach space. Let $\mathfrak{X}$ be a Banach space (real or complex).

In this case, Theorem 1 is not valid. However, the proof of Theorem 1 reveals the following fact.

Theorem 2. *Suppose that* (i) *holds and let* J_λ *be the corresponding resolvent. Then,* (ii) *holds if and only if*

(vii) $$\|\lambda J_\lambda - 1\| \leqq 1 \quad \text{for each } \lambda > 0.$$

If (ii) *holds, then the resolvent* J'_λ *of the semigroup in* (ii) *satisfies*

$$J'_\lambda = \frac{1}{\lambda}\left(1 - \frac{1}{\lambda} J_{1/\lambda}\right) \quad \text{and} \quad J_\lambda = \frac{1}{\lambda}\left(1 - \frac{1}{\lambda} J'_{1/\lambda}\right).$$

Another partial generalization of Theorem 1 is the following. Define

$$\tau(f, g) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}(\|f + \varepsilon g\| - \|f\|) \quad \text{and} \quad \tau'(f, g) = -\tau(f, -g)$$

for each pair of f and g in $\mathfrak{X}$.

Theorem 3. *Let* A *be a linear operator in* $\mathfrak{X}$. *Then, the following condition* (viii) *is equivalent to* (i), *while* (ix) *is equivalent to* (ii):

(viii) $\mathfrak{D}(A)$ *and* $\mathfrak{R}(A)$ *are both dense,* $\mathfrak{R}(1-A) = \mathfrak{X}$, *and*

(5) $$\tau'(f, Af) \leqq 0 \quad \text{for } f \in \mathfrak{D}(A).$$

(ix) $\mathfrak{D}(A)$ *and* $\mathfrak{R}(A)$ *are both dense,* $\mathfrak{R}(1-A) = \mathfrak{X}$, *and*

(6) $$\tau'(Af, f) \leqq 0 \quad \text{for } f \in \mathfrak{D}(A).$$

Proof. The equivalence of (i) and (viii) is essentially due to Hasegawa [1] (see [7] p. 439). We observe in [7] p. 440 that $\tau'(f, g) \leqq 0$ if and only if $\|\lambda f\| \leqq \|\lambda f - g\|$ for all $\lambda > 0$. Hence, the equivalence of (ii) and (ix) is essentially the result of Hirsch [2] p. 1487.

Theorem 3 remains valid if we replace $\tau'(f, g)$ by the semi-inner-product $[g, f]$ of Lumer-Phillips [5]. If $\mathfrak{X}$ is Hilbert, then $\tau'(f, g) = \mathrm{Re}\,(f, g)/\|f\|$ for $f \neq 0$, and hence each of the properties (5) and (6) is equivalent to dissipativeness. Thus Theorem 3 furnishes another proof of the equivalence of (i), (ii) and (iv) in Theorem 1.

Unlike the case of Hilbert space, any one of Conditions (viii) and (ix) does not imply the other in general. For example, let $\mathfrak{X} = C_0(-\infty, +\infty)$ be the space of real-valued continuous functions vanishing at $\pm\infty$ with the norm of uniform convergence, and let $Af = f'$ with $\mathfrak{D}(A) = \{f : f \text{ and } f' \in \mathfrak{X}\}$ or $Af = f''$ with $\mathfrak{D}(A) = \{f : f \text{ and } f'' \in \mathfrak{X}\}$. Then, A satisfies (viii) by [8]. But A does not satisfy (6), which is proved by using the expression of τ' in $C_0(-\infty, +\infty)$ in [6] p. 432. Also (ix) is valid with $-V = A^{-1}$ in place of A, while $\tau'(f, -Vf) \leqq 0$ does not hold.

Even if A satisfies (5) with domain and range both dense, and is maximal among such operators, it does not necessarily satisfy $\mathfrak{R}(1-A) = \mathfrak{X}$. An operator A in the space $\mathfrak{X} = C_0(0, +\infty)$ defined by $Af = f'$ with $\mathfrak{D}(A) = \{f : f \text{ and } f' \in \mathfrak{X}\}$ serves as an example for this. The proof is analogous to [4] p. 688.

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