9. *The Second Dual Space for the Space* N^+

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1. Introduction. Let D be the unit disk $\{|z|<1\}$. A holomorphic function $f(z)$ in D is said to belong to the *class* N of functions of bounded characteristic if

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta=O(1) \text{ as } r\to 1. \tag{1.1}$$

A function $f(z)\in N$ is said to belong to the *class* N^+ [2, p. 25] if

$$\lim_{r\to 1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta=\int_0^{2\pi}\log^+|f(e^{i\theta})|\,d\theta. \tag{1.2}$$

We showed in [7] that the class N^+ becomes an F-space in the sense of Banach [1, p. 51] with the distance function

$$\rho(f,g)=\frac{1}{2\pi}\int_0^{2\pi}\log(1+|f(e^{i\theta})-g(e^{i\theta})|)d\theta \tag{1.3}$$

The space N^+ with this metric (1.3) is not locally convex and not locally bounded [7, corollary to Theorem 2]. But N^+ has sufficiently many continuous linear functionals to form a dual system $<(N^+)^*, N^+>$ in the sense of Dieudonné and Mackey [5, p. 88].

Duren, Romberg, and Shields [3] studied the dual space $(H^p)^*$ of $H^p, 0<p<1$, and defined the containing Banach space $B^p\subset(H^p)^{**}$. Treating the corresponding problems for N^+, instead of H^p, we defined the containing Fréchet space F^+ for the class N^+ [8]. We will show in this note that F^+ is nothing but the second dual $(N^+)^{**}$ of N^+, and will obtain some results on its properties.

2. The space $(N^+)^{**}$. We denote by S the collection of complex sequences $\{b_n\}$ such that

$$\overline{\lim_{n\to\infty}}\{(1/\sqrt{n})\log^+|b_n|\}<0. \tag{2.1}$$

(2.1) means: there are constants $K=K(\{b_n\})$, $c=c(\{b_n\})>0$ such that

$$|b_n|\leqq K\exp[-c\sqrt{n}]. \tag{2.2}$$

In [7, Theorem 3] we showed:

Let ϕ be a continuous linear functional on N^+. Then there is a unique holomorphic function $g(z)=\sum b_n z^n$, continuous on $\bar{D}$, such that for any $f(z)=\sum a_n z^n\in N^+$

$$\begin{aligned}\phi(f)&=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})g(e^{-i\theta})d\theta\\&=\sum_{n=0}^{\infty}a_n b_n \text{ (absolutely convergent)}\end{aligned}\tag{2.3}$$

The Taylor coefficients $\{b_n\}$ *of the representing function* $g(z)$ *satisfies* (2.1). *Conversely, if* $\{b_n\}$ *satisfies* (2.1), *a continuous linear fuctional* ϕ *on* N^+ *is defined by* (2.3).

Hence, S *can be identified with the dual space of* N^+.

We defined in [8] a Fréchet space F^+ containing N^+. A function $f(z)$ belongs to F^+ if

$$\|f\|_{Fc}=\int_0^1 \exp\left[\frac{-c}{1-r}\right]M(r,f)dr<\infty \tag{2.4}$$

for any $c>0$, where

$$M(r,f)=\max_{|z|=r}|f(z)|. \tag{2.5}$$

$f(z)=\sum a_n z^n$ belongs to F^+ if and only if

$$M(r,f)=O(\exp[o(1)/(1-r)])$$

i.e.,

$$\overline{\lim_{r\to 1}}\{(1-r)\log M(r,f)\}\leqq 0, \tag{2.6}$$

or, equivalently,

$$a_n=O(\exp[o(\sqrt{n})])$$

i.e.,

$$\overline{\lim_{n\to\infty}}(1/\sqrt{n})\log|a_n|\leqq 0.$$

F^+ is endowed with the family of semi-norms $\{\|f\|_{Fc}\}_{c>0}$, which is equivalent to the family of semi-norms $\{\|f\|_c\}_{c>0}$, where (see [8])

$$\|f\|_c=\sum_{n=0}^{\infty}|a_n|\exp[-c\sqrt{n}] \qquad \text{for } f(z)=\sum a_n z^n. \tag{2.7}$$

N^+ is a dense subspace of F^+. We have shown in [8] that S *can also be identified with the dual space of* F^+. That is, if $\{b_n\}\in S$, a continuous linear functional ψ on F^+ is defined by

$$\psi(f)=\sum_{n=0}^{\infty}a_n b_n \text{ (absolutely convergent)} \qquad \text{for } f(z)=\sum_{n=0}^{\infty}a_n z^n.$$

Theorem 1. *Let* E *be a subset of* F^+. E *is bounded if and only if there is a constant* $A>0$ *and a sequence* $\{\lambda_n\}$, $\lambda_n\downarrow 0$, *such that*

$$|a_n|\leqq A\exp[\lambda_n\sqrt{n}] \qquad \text{for } f(z)=\sum a_n z^n\in E. \tag{2.8}$$

Proof. "If" part is obvious.

"Only if" part. Suppose E is bounded. Take a $c>0$ and let

$$V=\{g\in F^+;\|g\|_c<\eta\} \tag{2.9}$$

be a neighborhood of 0. There is an α such that $\alpha E\subset V$, hence

$$\sum_{n=0}^{\infty}|\alpha a_n|\exp[-c\sqrt{n}]<\eta, \qquad f(z)=\sum a_n z^n\in E,$$

therefore

$$|a_n|\leqq|\eta/\alpha|\exp[c\sqrt{n}].$$

Thus, if we put

$$a_n^*=\{\sup|a_n(f)|;\ f(z)=\sum a_n(f)z^n\in E\}$$

we get

$$a_n^*\leqq|\eta/\alpha|\exp[c\sqrt{n}],$$

hence

$$\lim_{n\to\infty} (1/\sqrt{n}) \log a_n^* \leqq c.$$

Since c is arbitrary, we know that $a_n^* = O(\exp[o(\sqrt{n})])$, and (2.8) holds. Q.E·D.

Since boundedness and weakly-boundedness coincide in F^+ [5, p. 255], we have obviously, from Theorem 1,

Theorem 2. *Let E be a subset of N^+. E is weakly bounded if and only if there is a constant $A>0$ and a sequence $\{\lambda_n\}$, $\lambda_n \downarrow 0$ such that*

$$|a_n| \leqq A \exp[\lambda_n \sqrt{n}] \quad \text{for } f(z) = \sum a_n z^n \in E. \tag{2.8'}$$

We denote by $(N^+)^*$ the space S with the topology of uniform convergence on weakly bounded subsets of N^+.

We also denote by $(F^+)^*$ the space S with the topology induced by F^+, i.e., the topology of uniform convergence on bounded subsets of F^+. Then, from Theorems 1 and 2, we obtain

Theorem 3. $(F^+)^* = (N^+)^*$. (both set-theoretically and topologically).

Next we have

Theorem 4. *The space F^+ is nuclear.*

Proof follows from the theorem of Grothendieck and Pietsch [6, p. 88, Theorem 6.1.2] and the definition of semi-norms (2.7).

Corollary 1. *F^+ is a Montel space.*

Proof is known by Theorem 4 and [6, p. 73, Theorem 4.4.7].

Corollary 2. *F^+ is reflexive. Hence $(F^+)^*$ is reflexive.*

Proof. Every Montel space is reflexive [5, p. 372] and the strong dual of a reflexive space is reflexive [5, p. 305(5)].

By Theorem 3 and Corollary 2, we get

Theorem 5. $(N^+)^{**} = (F^+)^{**} = F^+$.

3. Multipliers for F^+. Let X and Y be some collections of complex sequences. A sequence of complex numbers $M = \{\mu_n\}$ is said to be a *multiplier* for X into Y, denoted as $M \in (X, Y)$, if

$$\text{for any } f = \{a_n\} \in X, \text{ we have } M[f] = \{\mu_n a_n\} \in Y. \tag{3.1}$$

Multipliers for H^p or B^p have been studied by several authors, see for example [2, p. 99] or [4]. As an application of Theorem 1, we have

Theorem 6. *A sequence $M = \{\mu_n\}$ is a multiplier for F^+ into B^p, $0<p<1$, if and only if $\{\mu_n\}$ satisfies (2.1), i.e.,*

$$\mu_n = O(\exp[-c\sqrt{n}]) \tag{3.2}$$

for some constant $c>0$.

Proof. The multiplier operator $M = \{\mu_n\}$, which assigns to $f(z) = \sum a_n z^n \in F^+$ a function $M[f](z) = \sum \mu_n a_n z^n \in B^p$, is obviously linear and closed. Hence M is continuous, and maps bounded subsets of F^+ to bounded subsets of B^p.

We note that for $g(z) = \sum b_n z^n \in B^p$ there hold

$$|b_n| \leqq C \|g\|_{B^p} \times n^{1/p-1} \tag{3.3}$$

with a constant C [3, p. 41], where $\|g\|_{B^p}$ is the norm in the space B^p:

$$\|g\|_{B^p}=\int_0^1(1-r)^{1/p-2}dr\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|d\theta. \tag{3.4}$$

Using (3.3) and [7, Lemma 1], we get our result. Q.E.D.

4. The space $(N^+)^*$.

We have

Theorem 7. *$(N^+)^*=(F^+)^*$ is bornological.*

Proof. By Corollary 2, F^+ is reflexive Fréchet space. Hence its strong dual is bornological [5, p. 403(4)]. Q.E.D.

Finally we note the following

Theorem 8. *Let E^* be a subset of $(N^+)^*$. E^* is bounded if and only if there are constants $K=K(E^*)>0$ and $c=c(E^*)>0$ such that*

$$|b_n(\phi)|\leqq K\exp[-c\sqrt{n}] \tag{4.1}$$

for all $\phi=\{b_n(\phi)\}\in E^*$.

Proof proceeds in a similar way as in Theorem 1, with neighborhood

$$V=\{\phi\in(N^+)^*;\sup_{f\in E}|\phi(f)|<\eta\} \tag{4.2}$$

instead of (2.9), where E is a bounded subset of F^+. Q.E.D.

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