4. A Remark on Integral Equation in a Banach Space

By Kenji MARUO and Naoki YAMADA

(Comm. by Kôsaku YOSIDA, M. J. A., Jan. 12, 1973)

1. Introduction and main theorem.

The main object of this paper is to extend the result of G. Webb [1] on the solution of the integral equation associated with some nonlinear equation of evolution in a Banach space to the time dependent case.

Let E be a Banach space with norm $\| \ \|$.

Let $A(t)$ $(0 \leqq t \leqq T)$ be a linear accretive operator which satisfies the conditions of T. Kato [2], H. Tanabe [3] or T. Kato and H. Tanabe [4], and $B(t)$ be a nonlinear, accretive, everywhere defined operator such that $(t, u) \to B(t)u$ is a strongly continuous mapping from $[0, T] \times E$ to E which maps bounded sets to bounded sets. It is known that there exists an evolution operator $U(t, \tau)$ $0 \leqq \tau \leqq t \leqq T$ with norm $\leqq 1$ to the linear equation $du(t)/dt + A(t)u(t) = 0$, and that $A(t)$ is m-accretive for $t \in [0, T]$.

Then we can state our main theorem.

Theorem. *Under our assumption, for any $x \in E$, $\tau \in [0, T[$, there exists a unique solution $u(t, \tau; x)$ to the integral equation*

$$(E) \qquad u(t, \tau; x) = U(t, \tau)x - \int_\tau^t U(t, s)B(s)u(s, \tau; x)ds$$

on $[\tau, T]$. If we define $W(t, \tau)x = u(t, \tau; x)$, then $W(t, \tau)$ has the following evolution properties and an inequality,

$$(1) \qquad W(t, \tau) = W(t, t')W(t', \tau), \quad W(t, t) = I \quad \text{for } 0 \leqq \tau \leqq t' \leqq t \leqq T$$

$$(2) \qquad W(t, \tau)x \text{ is strongly continuous in } 0 \leqq \tau \leqq t \leqq T$$

$$(3) \qquad \| W(t, \tau)x - W(t, \tau)y \| \leqq \| x - y \|$$

The authors wish to thank Professor H. Tanabe for his advices.

2. Proof of the theorem.

The main idea of the proof is due to G. Webb [1].

Proposition 1. *For any $x \in E$, $\tau \in [0, T[$, there exists $T_0 (\tau < T_0 \leqq T)$ and a continuous function $u(t, \tau; x): [\tau, T_0] \to E$ such that $u(t, \tau; x)$ is a solution of (E) on $[\tau, T_0]$.*

Proof. Let $x \in E$, $\tau \in [0, T[$ be fixed. In view of the continuity of $B(t)x$, for any $\varepsilon > 0$ there exists a positive number δ depending on x, τ, ε, such that for any $v \in V = \{v : \| x - v \| < \delta\}$ and any $t, |t - \tau| < \delta$ the inequality $\|B(t)v - B(\tau)x\| \leqq \varepsilon$ hold. Take $M = \|B(\tau)x\| + \varepsilon$ then $\|B(t)v\| \leqq M$ for any $v \in V$ and $t, |t - \tau| < \delta$. Under the assumptions of [2] or [3] we take the sequence $x_n \in D(A(t))$ such that x_n converges to x, and in case of [4] we put $x_n = x$. We write $v = U(t,\tau)x_n + \omega$. Then we can choose $T_1 > \tau$ and a large positive integer N such that v are points in V for any integer $n \geqq N$, any number t; $\tau \leqq t \leqq T_1$ and any point $\omega \in E$; $\|\omega\| \leqq (T_1 - \tau)M$. Let $T_0 = \mathrm{Min}\{T_1, \tau + \delta\}$. For any positive integer $n \geqq N$, let $t_0^n = \tau$, $u_n(t_0^n) = x_n$. Inductively, for each positive integer i, define δ_i^n, t_i^n, and $u_n(t_{i-1}^n)$ such that

(i) $0 \leqq \delta_i^n$, $t_{i-1}^n + \delta_i^n \leqq T_0$

(ii) if $\|z - u_n(t_{i-1}^n)\| \leqq \delta_i^n M + \mathrm{Max}_{t_{i-1}^n \leqq t \leqq t_{i-1}^n + \delta_i^n} \|[U(t, t_{i-1}^n) - I]u_n(t_{i-1}^n)\|$ then $\mathrm{Sup}_{t_{i-1}^n \leqq t \leqq t_{i-1}^n + \delta_i^n} \|B(t)z - B(t_{i-1}^n)u_n(t_{i-1}^n)\| \leqq 1/n$ and δ_i^n is the largest number such that (i) and (ii) hold.

Define $t_i^n = t_{i-1}^n + \delta_i^n$ and for each $t \in [t_{i-1}^n, t_i^n]$ define

$$u_n(t) = U(t, t_{i-1}^n)u_n(t_{i-1}^n) - \int_{t_{i-1}^n}^{t} U(t,s)B(t_{i-1}^n)u_n(t_{i-1}^n)ds. \tag{2.1}$$

It is easy to see that for $t \in [t_{k-1}^n, t_k^n]$

$$u_n(t) = U(t,\tau)x_n - \sum_{i=1}^{k-1} \int_{t_{i-1}^n}^{t_i^n} U(t,s)B(t_{i-1}^n)u_n(t_{i-1}^n)ds - \int_{t_{k-1}^n}^{t} U(t,s)B(t_{k-1}^n)u_n(t_{k-1}^n)ds. \tag{2.2}$$

By the same argument as G. Webb [1], we see that $u_n(t) \in V \cap D(A(t))$ and

$$\mathrm{Sup}_{t_{i-1}^n \leqq t \leqq t_i^n} \|B(t)u_n(t) - B(t_{i-1}^n)u_n(t_{i-1}^n)\| \leqq 1/n \tag{2.3}$$

by the estimate of $\|u_n(t) - u_n(t_{i-1}^n)\|$ and (2.1).

If $t \in]t_{i-1}^n, t_i^n[$, $u_n(t)$ is differentiable at t and

$$u_n'(t) = -(A(t)u_n(t) + B(t_{i-1}^n)u_n(t_{i-1}^n)). \tag{2.4}$$

We will show that there exists some positive integer L such that $t_L^n = T_0$. Assume that $t_i^n < T_0$ for all i. Following the same method as [1] we see that $\lim_{i\to\infty} u_n(t_i^n)$ exists. Let $z_0 = \lim_{i\to\infty} u_n(t_i^n)$ and $t_0 = \lim_{i\to\infty} t_i^n$. Choose $\alpha > 0$ and $\beta_0 > 0$ such that if $\|z - z_0\| < \alpha$, $|t - t_0| < \beta_0$ then $\|B(t)z - B(t_0)z_0\| < 1/4n$. Noting that $\{u_n(t_i^n)\}_{i=1}^{\infty}$ is compact there exists $\beta_1 > 0$ such that if $t_{i-1}^n \leqq t \leqq t_{i-1}^n + \beta_1$, then $\|[U(t, t_{i-1}^n) - I]u_n(t_{i-1}^n)\| < \alpha/4$ for all i. Let $\beta = \mathrm{Min}\{\beta_0, \beta_1\}$ and choose k so large that

$$\delta_k^n < \alpha/4M, \delta_k^n < \beta, \quad \|u_n(t_{k-1}^n) - z_0\| < \alpha/4 \quad \text{and} \quad t_0 - \beta < t_{k-1}^n.$$

If

$$\|z - u_n(t_{k-1}^n)\| \leqq \delta_k^n M + \mathrm{Max}_{t_{k-1}^n \leqq t \leqq t_k^n} \|[U(t, t_{k-1}^n) - I]u_n(t_{k-1}^n)\| + \alpha/4$$

then arguing as in Webb [1] we know

$$\|B(t)z - B(t_{k-1}^n)u_n(t_{k-1}^n)\| \leqq \|B(t)z - B(t_0)z_0\| + \|B(t_0)z_0 - B(t_{k-1}^n)u_n(t_{k-1}^n)\| \leqq 1/2n.$$

This contradicts the definition of δ_k^n, so there exists some integer L such that $t_L^n = T_0$. Next we will show that continuous function $u_n(t)$ converges uniformly on $[\tau, T_0]$. Define $P_{n,m}(t) = \|u_n(t) - u_m(t)\|$ and let $t \in]\tau, T_0[$ be such that $t \in]t_{j-1}^m, t_j^m[$ and $t \in]t_{k-1}^n, t_k^n[$ for some integer j, k. In view of (2.3) and (2.4)

$$P_{n,m}^{-\prime}(t) \leqq \lim_{h \uparrow 0} 1/h\{\|u_n(t)-u_m(t)-h[(A(t)+B(t))u_n(t) - (A(t)+B(t))u_m(t)]\| - \|u_n(t)-u_m(t)\|\} + \|B(t)u_n(t)-B(t_{k-1}^n)u_n(t_{k-1}^n)\| + \|B(t)u_m(t)-B(t_{j-1}^m)u_m(t_{j-1}^m)\| \leqq 1/n+1/m.$$

Here we used the accretiveness of $A(t)+B(t)$. Hence we have

$$P_{n,m}(t) \leqq \|x_n - x_m\| + (T_0-\tau)(1/n+1/m)$$

and so $u_n(t)$ converges uniformly to a continuous function $u(t,\tau\,;x)$. From (2.3) and noting that $B(s)u_n(s)$ converges to $B(s)u(s,\tau\,;x)$ for each s as $n\to\infty$ and $\|B(s)u_n(s)\|\leqq M$ for $s\in[\tau, T_0]$, using Lebesgue's theorem, we see that $u(t,\tau\,;x)$ satisfies the equation (E) on $[\tau, T_0]$.

Proposition 2. *Let $u(t,\tau\,;x)$ and $v(t,\tau\,;y)$ be the solutions of (E) on $[\tau, T_1]$ and $[\tau, T_2]$ for any $x, y\in E$, respectively. Then we find*

(2.5) $$\|u(t,\tau\,;x)-v(t,\tau\,;y)\| \leqq \|x-y\|$$

for any $t\,;\tau\leqq t\leqq \min\{T_1, T_2\}$. Consequently the solution of (E) is unique and satisfies the relation

(2.6) $$u(t,\tau\,;x)=u(t,t'\,;u(t',\tau\,;x))$$

for t and $t'\,;\tau\leqq t'\leqq t\leqq T_1$.

Proof. Take sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ as in the proof of Proposition 1. Let $\{t_i^n\}_{i=0}^n$ be a partition of $[\tau, \min\{T_1, T_2\}]$ for each n. Define for $t\in[t_{k-}^n, t_k^n]$

$$u_n(t\,;x)=U(t,\tau)x_n-\sum_{i=1}^{k-1}\int_{t_{i-1}^n}^{t_i^n} U(t,s)B(t_{i-1}^n)u(t_{i-1}^n,\tau\,;x)ds - \int_{t_{k-1}^n}^{t} U(t,s)B(t_{k-1}^n)u(t_{k-1}^n,\tau\,;x)ds$$

and $v_n(t,y)$ similarly. It is easy to see that $u_n(t\,;x)$ and $v_n(t\,;y)$ are differentiable for $t\in\,]t_{k-1}^n, t_k^n[$ and

$$u_n'(t\,;x)=-(A(t)u_n(t\,;x)+B(t_{k-1}^n)u(t_{k-1}^n,\tau\,;x))$$

and similarly for $v_n(t\,;y)$. Furthermore $u_n(t\,;x)$ and $v_n(t\,;y)$ converge uniformly to $u(t,\tau\,;x)$ and $v(t,\tau\,;y)$ respectively as the mesh of $\{t_i^n\}$ goes to zero with n. Let $P_n(t)=\|u_n(t\,;x)-v_n(t\,;y)\|$. By the same argument as in Proposition 1, we obtain

(2.7) $$P_n^{-\prime}(t)\leqq\|B(t)u_n(t\,;x)-B(t_{k-1}^n)u(t_{k-1}^n,\tau\,;x)\| + \|B(t)v_n(t\,;y)-B(t_{k-1}^n)v(t_{k-1}^n,\tau\,;y)\|.$$

Using Lebesgue's theorem we obtain

$$\lim_{n\to\infty} P_n(t)\leqq\|x-y\|.$$

Hence the uniqueness of the solution follows at once. On the other hand we know (2.6) from the uniqueness of the solution.

Proposition 3. *For any $x\in E$, $\tau\in[0,T[$ the solution $u(t,\tau\,;x)$ of (E) exists on $[\tau, T]$.*

Proof. Assume that $u(t,\tau\,;x)$ exists on $[\tau, T_0[$ for some $T_0\leqq T$. First we will show that

$$\operatorname*{Sup}_{\tau\leqq t\leqq T_0}\|u(t,\tau\,;x)\|\leqq C$$

where C is a constant which depends only T, B and x. Let T' be fixed such that $\tau < T' < T_0$. On $[\tau, T']$ we define the approximating function $u_n(t;x)$ as in the proof of Proposition 2 and define $P_n(t) = \|u_n(t;x)\|$. Then we find

$$P_n^{-\prime}(t) \leqq \|B(t)0\| + \|B(t)u_n(t;x) - B(t_{k-1}^n)u(t_{k-1}^n, \tau; x)\|$$

as Proposition 2 and so for $t \in [t_{k-1}^n, t_k^n]$

$$\begin{aligned} P_n(t) \leqq \|x_n\| &+ \int_\tau^t \|B(s)0\|\,ds \\ &+ \sum_{i=1}^{k-1} \int_{t_{i-1}^n}^{t_i^n} \|B(s)u_n(s;x) - B(t_{i-1}^n)u(t_{i-1}^n, \tau; x)\|\,ds \\ &+ \int_{t_{k-1}^n}^{t} \|B(s)u_n(s;x) - B(t_{k-1}^n)u(t_{k-1}^n, \tau; x)\|\,ds. \end{aligned}$$

The third and fourth terms tend to zero as the mesh goes to zero with $n \to \infty$, and hence we obtain

$$\|u(t, \tau; x)\| \leqq \|x\| + \int_0^T \|B(s)0\|ds$$

on $[\tau, T']$, but the right hand side is independent of T'. So we obtain the boundedness of $u(t, \tau; x)$ on $[\tau, T_0[$. Let $h, h' > 0$, $h - h' \geqq 0$, $T_0 - h \geqq \tau$ and let us estimate $\|u(T_0 - h, \tau; x) - u(T_0 - h', \tau; x)\|$. Using the assumption on $B(t)$ and the boundedness of $u(t, \tau; x)$ just shown, we see that $\lim_{t \uparrow T_0} u(t, \tau; x)$ exists and so $u(t, \tau; x)$ can be continued past T_0.

Proposition 4. *Define $W(t, \tau)x = u(t, \tau; x)$, then $W(t, \tau)x$ satisfies the properties stated in the theorem.*

Proof. It remains only to prove the continuity of $W(t, \tau)$. Let $\tau \leqq \tau' \leqq t$ then from (2.5), (2.6)

$$\|u(t, \tau; x) - u(t, \tau'; x)\| \leqq \|u(\tau', \tau; x) - x\|.$$

Hence $u(t, \tau; x)$ is continuous in τ and t: $0 \leqq \tau \leqq t \leqq T$. So the theorem is proved.

References

[1] Webb, G.: Continuous nonlinear perturbations of linear accretive operators in Banach spaces. J. Funct. Anal., **10**, 191–203 (1972).

[2] Kato, T.: Integration of the equation of evolution in a Banach space. J. Math. Soc. Japan, **5**, 208–234 (1953).

[3] Tanabe, H.: On the equations of evolution in a Banach space. Osaka Math. J., **12**, 363–376 (1960).

[4] Kato, T., and Tanabe, H.: On the abstract evolution equation. Osaka Math. J., **14**, 107–133 (1962).