# 26. On Some Examples of Non-normal Operators. III

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**1. Introduction.** In the previous note [3 ; II], we have introduced the hen-spectra of operators. If $T$ is an operator acting on a Hilbert space $\mathfrak{H}$ with the spectrum $\sigma(T)$, then the *hen-spectrum* $\tilde{\sigma}(T)$ is the complement of the unbounded component of $\sigma(T)^c$ where $M^c$ is the complement of a set $M$ in the complex plane. Clearly, the hen-spectrum is a compact set in the plane with the connected complement, and we have proved in [3 ; II, Proposition 2].

$$(1) \qquad \sigma(T) \subset \tilde{\sigma}(T) \subset \mathrm{co}\,\sigma(T) \subset \overline{W}(T),$$

where $\mathrm{co}\,M$ is the convex hull of $M$, $\overline{M}$ the closure of $M$, and $W(T)$ is the numerical range of $T$.

In the previous note [3 ; II], we are concerned with growth conditions: An operator $T$ is called to satisfy the *condition* $(G_1)$ (resp. $(H_1)$) if

$$(2) \qquad \|(T-\lambda)^{-1}\| \leqq \frac{1}{\mathrm{dist}\,(\lambda, X)}$$

for $\lambda \notin X$ and $X=\sigma(T)$ (resp. $X=\tilde{\sigma}(T)$). By (2), we have, $T \in (G_1)$ implies $T \in (H_1)$, and $T \in (H_1)$ implies that $T$ is a convexoid in the sense of Halmos [5], i.e. $\overline{W}(T)=\mathrm{co}\,\sigma(T)$.

In the present note, we shall concern with spectral sets introduced by von Neumann: A closed set $S$ in the complex plane called a *spectral set for an operator* $T$ if

$$(3) \qquad \sigma(T) \subset S$$

and

$$(4) \qquad \|f(T)\| \leqq \|f\|_S,$$

where $f$ is a rational function with poles off $S$ and

$$\|f\|_S = \sup_{z \in S} |f(z)|,$$

cf. [6] for details. If $S$ is a spectral set for $T$ and $S \subset S'$, then $S'$ is also a spectral set for $T$. A fundamental theorem for spectral set is

**Theorem A** (von Neumann). *The (closed) unit disk $D$ is a spectral set for every contraction.*

The following theorem, also due to von Neumann, is a direct consequence of Theorem $A$:

**Theorem B.** $\{\alpha ; |\alpha-\lambda| \geqq \beta\}$ *is a spectral set for $T$ if and only if* $\|(T-\lambda)^{-1}\| \leqq 1/\beta$.

The following theorem obtained in [6] is a principal tool in the below:

**Theorem C** (Lebow). *If $S$ is a compact set which does not separate the plane, then $S$ is a spectral set for an operator $T$ if and only if*

$$(4') \qquad \|p(T)\| \leqq \|p\|_S$$

*for any polynomial $p$.*

In the below, we shall study a class of non-normal operators defined by spectral sets. We shall introduce a new class of operators and discuss some properties in § 2. Following after [4], we shall construct an example in § 3. Inclusion relations of classes of non-normal operators are discussed in § 4. In §§ 5–6, we shall give two characterizations of new class in terms of dilations and polynomials of operators. In § 7, we make two remarks.

**2. Definition.** By means of spectral sets, Hildebrandt [4] introduced two classes of non-normal operators: $T$ is a *spectroid* (resp. *numeroid*, in the sense of [3 ; I]) if $\sigma(T)$ (resp. $\overline{W}(T)$) is a spectral set for $T$. In this direction, we introduce

**Definition 1.** An operator $T$ is a *hen-spectroid* if $\tilde{\sigma}(T)$ is a spectral set for $T$.

We shall list up some elementary properties of hen-spectroids:

**Proposition 2.** *A spectroid is a hen-spectroid; and a hen-spectroid is a numeroid.*

**Proof.** By the definitions, (2) implies the proposition.

**Proposition 3.** *A hen-spectroid satisfies $(H_1)$.*

**Proof.** If $\lambda \notin \tilde{\sigma}(T)$ and

$$\tilde{\sigma}(T) \subset \{\alpha ; |\alpha-\lambda| \geqq \beta\}$$

for $\beta > 0$, then we have

$$\|(T-\lambda)^{-1}\| \leqq \frac{1}{\beta}$$

by Theorem B. Hence we have $T \in (H_1)$.

**Proposition 4.** *$T$ is a hen-spectroid if and only if (4′) is satisfied for any polynomial $p$ for $S=\tilde{\sigma}(T)$.*

**Proof.** If $T$ is a hen-spectroid, then we have (4′) for $S=\tilde{\sigma}(T)$. Conversely, if (4′) is satisfied for any polynomial $p$, then $\tilde{\sigma}(T)$ is a spectral set for $T$ by Theorem C since $\tilde{\sigma}(T)^c$ is connected.

**Proposition 5.** *A compact hen-spectroid is normal.*

**Proof.** If $T$ is compact, then $\sigma(T)$ is at most countable, so that $\sigma(T)^c$ is connected, and we have $\sigma(T)=\tilde{\sigma}(T)$. Hence $\sigma(T)$ is a spectral set for $T$ by the hypothesis, or $T$ is a spectroid. It is well-known that a compact spectroid is normal.

**3. Construction.** In this section, we shall give a method to construct a hen-spectroid:

**Theorem 6.** *For an arbitrary operator $A$ with a compact spectral set $S$, there is a normal operator $B$ with $S=\sigma(B)$ such that $S$ is a spectral set for $T=A\oplus B$.*

**Proof.** If $f$ is a rational function with poles off $S$, then we have

$$\|f(T)\|=\|f(A\oplus B)\|=\|f(A)\oplus f(B)\|$$
$$=\max(\|f(A)\|, \|f(B)\|)\leqq\|f\|_S$$

since the spectrum is a spectral set for a normal operator. Hence $S$ is a spectral set $T$.

**Corollary 7.** *For any $A$, there is a normal operator $B$ such that $T=A\oplus B$ is a hen-spectroid.*

**Proof.** By Theorem 6, $S=\tilde{\sigma}(A)$ is a spectral set for $T$. Since $\sigma(T)=\sigma(A)\cup\sigma(B)=S$, we have $\tilde{\sigma}(T)\supset S$, and $\tilde{\sigma}(T)$ is a spectral set for $T$, or $T$ is a hen-spectroid.

**Remark.** In the previous note [3; I, Theorem 3], we have constructed a numeroid by a similar method, assuming $S\subset\overline{W}(B)$. However, this is insufficient: We need to assume that $S\cup\overline{W}(A)\subset\overline{W}(B)$, so that we can prove that $\overline{W}(T)=\mathrm{co}\{\overline{W}(A), \overline{W}(B)\}=\overline{W}(B)$ is a spectral set for $T$.

**4. Application.** We shall prove

**Theorem 8.** *There is a hen-spectroid which does not satisfy $(G_1)$.*

**Proof.** Let

$$A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

and $B$ be a simple bilateral shift. Then we have

$$\|A\|\leqq 1, \quad \|B\|=1, \quad \sigma(B)=C, \quad \tilde{\sigma}(B)=D,$$

where $C$ is the unit circle and $D$ the unit disk. By Theorem A, $D$ is a spectral set for $A$. Hence, by Corollary 7, $T=A\oplus B$ is a hen-spectroid and $\sigma(T)=\{0\}\cup C$. We have

$$\left(A+\frac{1}{2}\right)^{-1}=2\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix},$$

so that for

$$x=\begin{pmatrix}0\\ 1\end{pmatrix}$$

we have

$$\left\|\left(A+\frac{1}{2}\right)^{-1}\right\|\geqq\left\|\left(A+\frac{1}{2}\right)^{-1}x\right\|=2\sqrt{4+1}>2.$$

If $T\in(G_1)$, then we have

$$2<\left\|\left(A+\frac{1}{2}\right)^{-1}\right\|\leqq\left\|\left(T+\frac{1}{2}\right)^{-1}\right\|\leqq\frac{1}{\operatorname{dist}\left(-\frac{1}{2}, \sigma(T)\right)}=2,$$

which is a contradiction.

**Corollary 9.** *The class of all spectroids is properly contained in* *the class of all hen-spectroids.*

**Proof.** If not, then every hen-spectroid $T$ is a spectroid, so that $T \in (G_1)$, which is impossible by Theorem 8.

**Theorem 10.** *There is a numeroid which is not a hen-spectroid.*

**Proof.** We have proved in [3; II, Prop. 10], there is a numeroid which is not $(H_1)$. Hence Proposition 3 implies the theorem.

**Remark.** The converse of Theorem 8 is also valid: *There is $T \in (G_1)$ which is not a hen-spectroid.* If not, every $T \in (G_1)$ is a normaloid, which is impossible.

**5. Dilation.** For an operator $T$ acting on $\mathfrak{H}$, if there is normal operator $N$ acting on $\mathfrak{K}$ including $\mathfrak{H}$ which satisfies

$$T^n x = PN^n x \qquad (n=0,1,2,\cdots) \tag{5}$$

for $x \in \mathfrak{H}$, where $P$ is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then $N$ is called a *strong normal dilation* of $T$. The following theorem is basic in our study, cf. [5], [8] and [9]:

**Theorem D** (Berger-Foias-Lebow). *If $S$ is a (compact) spectral set for $T$, then there is a strong normal diation $N$ of $T$ with*

$$\sigma(N) \subset \partial S \tag{6}$$

*where $\partial S$ is the boundary of $S$.*

For numeroids, the following characterization is proved in [8]:

**Theorem E** (Schreiber). *An operator $T$ is a numeroid if and only if there is a strong normal dilation $N$ of $T$ with*

$$\overline{W}(N) = \overline{W}(T). \tag{7}$$

Schreiber's theorem suggests us the following characterizatins of spectroids and hen-spectroids:

**Theorem 11.** *$T$ is a hen-spectroid if and only if there is a strong normal dilation $N$ of $T$ with*

$$\tilde{\sigma}(N) \subset \tilde{\sigma}(T). \tag{8}$$

**Proof.** If $T$ is a hen-spectroid, then we have a strong normal dilation $N$ with (8) by Theorem $D$ taking $S = \tilde{\sigma}(T)$.

Conversely, if $N$ and $T$ satisfy the hypothesis of Theorem 11, then we have

$$\|p(T)\| \leqq \|p(N)\| \leqq \|p\|_{\tilde{\sigma}(N)} \leqq \|p\|_{\tilde{\sigma}(T)}$$

for any polynomial $p$ since we have $p(T)x = Pp(N)x$ for $x \in \mathfrak{H}$ by (5). Hence $T$ is a hen-spectroid by Proposition 4.

**Theorem 12.** *$T$ is a spectroid if and only if there is a strong normal dilation $N$ of $T$ with*

$$\sigma(N) \subset \partial\sigma(T). \tag{9}$$

**Proof.** If $T$ is a spectroid, then we have a strong normal dilation $N$ of $T$ with (9) by Theorem E taking $S = \sigma(T)$.

The converse is essentially same with the proof of Schreiber's theorem [8]. Using the Neumann expansion, we have

$$((T-\lambda)^{-1}x \mid y)=((N-\lambda)^{-1}x \mid y)$$

for any $\lambda \notin \sigma(T)$ and $x, y \in \mathfrak{H}$. Hence we have

$$(f(T)x \mid y)=(f(N)x \mid y)$$

for every rational function $f$ with poles off $\sigma(T)$. Therefore we have

$$\|f(T)\| \leqq \|f(N)\| \leqq \|f\|_{\sigma(N)} \leqq \|f\|_{\sigma(T)},$$

so that $\sigma(T)$ is a spectral set for $T$, or $T$ is a spectroid.

**6. Transposition.** Following after [5], we shall call an operator $T$ is a *normaloid* if $\|T\|=r(T)$ where $r(T)$ is the spectral radius of $T$. In [1], the following characterization of spectroids is proved:

**Theorem F** (Berberian). *$T$ is a spectroid if and only if $f(T)$ is a normaloid whenever $f$ is a rational function with poles off $\sigma(T)$.*

Inspired by Berberian's theorem, we shall give here a characterization of hen-spectroids:

**Theorem 13.** *$T$ is a hen-spectroid if and only if $p(T)$ is a normaloid for any polynomial $p$.*

**Proof.** At first, we state

$$r(p(T))=\|p\|_{\sigma(T)}=\|p\|_{\tilde{\sigma}(T)}, \tag{10}$$

for every polynomial $p$; because

$$\begin{aligned} r(p(T)) &= \sup\{|\mu| ; \mu \in \sigma(p(T))\} \\ &= \sup\{|\mu| ; \mu \in p(\sigma(T))\} \\ &= \sup\{|p(\lambda)| ; \lambda \in \sigma(T)\} \\ &= \|p\|_{\sigma(T)} \end{aligned}$$

by the spectral mapping theorem and

$$\|p\|_{\sigma(T)}=\|p\|_{\tilde{\sigma}(T)}$$

by the maximum modulus principle.

If $p(T)$ is a normaloid for every $p$, then (10) gives us

$$\|p(T)\|=r(p(T))=\|p\|_{\tilde{\sigma}(T)}$$

which tells us that $T$ is a hen-spectroid by Proposition 4.

Conversely, if $T$ is a hen-spectroid, then we have

$$\|p(T)\| \leqq \|p\|_{\tilde{\sigma}(T)}=r(p(T)) \leqq \|p(T)\|.$$

Hence we have $\|p(T)\|=r(p(T))$, so that $p(T)$ is a normaloid for every polynomial $p$.

**Remark.** Theorem 13 is a generalization of a theorem of Williams [10]: $T$ is a numeroid if $p(T)$ is a normaloid for any polynomial $p$. A similar proof for Theorem 13 is also obtained by R. Nakamoto in his private letter.

A similar proof for Theorem 13 given us that $T$ is a hen-spectroid if and only if (4′) is satisfied for every polynomial $p$ and $S=\sigma(T)$.

**7. Appendix.** In the previous note [3: II, § 4], we have defined a class $Q$ of operators: $T \in Q$ if

$$\tilde{\sigma}(T)=\operatorname{co}\sigma(T). \tag{11}$$

We have shown that the intersection of $Q$ and the class of all convexoids is $\mathcal{R}$ introduced by Luecke [7]. We have also proved, in [3; II, Theorem 3], $T \in \mathcal{R}$ if and only if

(12) $$\overline{W}(T) = \tilde{\sigma}(T).$$

In this section, we shall give two remarks on hen-spectroids with $Q$ and hyponormality. By a theorem of [4] and (12), we have

**Proposition 14.** *If $T \in Q$ is a numeroid, then $T$ is a hen-spectroid.*

**Proposition 15.** *There is a hyponormal operator which is not a hen-spectroid.*

**Proof.** Clancey's example in [2] presents us a hyponormal operator $T$ which is not a spectroid. However, his example satisfies that $\sigma(T)^c$ is connected. Hence $\sigma(T) = \tilde{\sigma}(T)$ and $T$ is not a hen-spectroid.

Finally, we shall prove the following characterization of a class of operators:

**Proposition 16.** *$T \in \mathcal{R}$ is a hen-spectroid if and only if there is a strong normal dilation $N$ of $T$ with $\overline{W}(N) = \tilde{\sigma}(T)$.*

**Proof.** If $T$ is a hen-spectroid, then $T$ is a numeroid, so that there is a strong normal dilation $N$ of $T$ with $\overline{W}(N) = \overline{W}(T)$ by Schreiber's theorem. Since $T \in \mathcal{R}$, we have $\overline{W}(N) = \overline{W}(T) = \tilde{\sigma}(T)$ by (12).

Conversely, if $\overline{W}(N) = \tilde{\sigma}(T)$ by a strong normal dilation $N$ of $T$, then $T$ is a hen-spectroid by Theorem 11. Moreover, we have

$$\tilde{\sigma}(T) \subset \overline{W}(T) \subset \overline{W}(N) = \tilde{\sigma}(T),$$

so that $T$ satisfies (12) and $T \in \mathcal{R}$.

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