21. On the Boundedness of a Class of Operator-valued Pseudo-differential Operators in L^p Space

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Introduction. In this paper we present a class of pseudo-differential operators which are continuous in $L^p(\boldsymbol{R}^n)$, $1<p<\infty$. They will play an important role in studying the complex interpolation spaces of Sobolev spaces (see [3]).

Our main tools are the operator-valued version of Calderón-Vaillancourt's L^2-boundedness theorem ([2]), the Marcinkiewicz interpolation theorem, and the real-variable technique of Calderón and Zygmund which gives the weak-type estimate.

Notations. $\mathcal{L}(X, Y)$—the space of bounded linear operators from a Banach space X to a Banach space Y.

$L^p(E, d\mu; X)$—the space of X-valued L^p functions on a measure space $(E, d\mu)$

$$L^p(\boldsymbol{R}^n; X)=L^p(\boldsymbol{R}^n, dx; X), \qquad L^p(E, d\mu)=L^p(E, d\mu; \boldsymbol{C}).$$

$$x=(x_1, \cdots, x_n) \in \boldsymbol{R}^n, \qquad \alpha=(\alpha_1, \cdots, \alpha_n), \alpha_j \text{ are integers,}$$

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad |\alpha|=\alpha_1+\cdots+\alpha_n,$$

$$|x|^2=x_1^2+\cdots+x_n^2, \quad D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \quad D_j=\partial/\partial x_j.$$

$\mathcal{S}(\boldsymbol{R}^n; X)$—the space of X-valued rapidly decreasing C^∞ functions.

$m(S)$—measure of the set $S\subset\boldsymbol{R}^n$. $a_n=m\{x\,|\,|x|\leqq 1\}$.

Definition. Let X, Y be two Banach spaces. Then an $\mathcal{L}(X, Y)$-valued infinitely differentiable function $p(x, \xi, y)$ of $(x, \xi, y)\in\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ belongs to $S^\mu_{\rho,\delta,\varepsilon}(\boldsymbol{R}^{3n}, X; Y)$ if

$$(1) \qquad \|D_x^\alpha D_\xi^\beta p(x, \xi, y)\|_{\mathcal{L}(X,Y)}\leqq C(1+|\xi|)^{\mu+\delta|\alpha|-\rho|\beta|},$$

$$(2) \qquad \|D_y^\gamma D_\xi^\beta p(x, \xi, y)\|_{\mathcal{L}(X,Y)}\leqq C(1+|\xi|)^{\mu+\varepsilon|\gamma|-\rho|\beta|},$$

for any multi-index α, β, γ, where $0\leqq\rho, \delta, \varepsilon\leqq 1$.

For any p of this kind with $\varepsilon<1$ and for any $f\in\mathcal{S}(\boldsymbol{R}^n; X)$ the integral

$$Tf(x)=\frac{1}{(2\pi)^n}\int e^{ix\xi}d\xi\int p(x, \xi, y)f(y)e^{-i\xi y}dy$$

$$=\frac{1}{(2\pi)^n}\int(1+|\xi|^2)^{-m}(1-\Delta_y)^m\{p(x, \xi, y)f(y)\}e^{i\xi(x-y)}d\xi dy$$

is well defined and Tf belongs to $\mathcal{S}(\boldsymbol{R}^n; Y)$, where m is a positive integer such that $2m(1-\varepsilon)>\mu+n$, and Δ_y the Laplacian operator.

Theorem 1. *Let X, Y be two Hilbert spaces,*

$$p(x, \xi, y) \in S^{\mu}_{\rho,\delta,\varepsilon}(\boldsymbol{R}^{3n}, X; Y), \qquad 0 \leqq \delta, \varepsilon < 1, 0 \leqq \rho \leqq 1,$$

and let $-2\mu \geqq n\{\max(\delta, \rho) + \max(\varepsilon, \rho)\} - 2n\rho$. *Then*

$$\|Tf\|_{L^2(\boldsymbol{R}^n; Y)} \leqq C\|p\| \cdot \|f\|_{L^2(\boldsymbol{R}^n; X)},$$

where C depends only on $\delta, \varepsilon, \rho, n$. Here $\|p\|$ denotes the least value of C for which (1) *and* (2) *hold for $|\alpha| \leqq 2m_1$, $|\beta| \leqq 2m$, $|\gamma| \leqq 2m_2$, where m, m_1, m_2 are the least integers such that $2m \geqq n+2$, $m_1(1-\delta') > 5n/4$, $m_2(1-\varepsilon') > 5n/4$, $\rho' = \min(\rho, \max(\delta, \varepsilon))$, $\delta' = \max(\delta, \rho')$, $\varepsilon' = \max(\varepsilon, \rho')$.*

Proof. Noting that p belongs to $S^{\mu}_{\rho',\delta',\varepsilon'}$, the theorem can be proved in the same way as Calderón-Vaillancourt [2], in which we shall need the following lemma:

Lemma 1. *Let X, Y be Hilbert spaces, and let $T(\sigma)$ be a strongly measurable, uniformly bounded $\mathcal{L}(X, Y)$-valued function on a measure space $(E, d\sigma)$ such that*

$$\|T(\sigma_1)^* T(\sigma_2)\|_{\mathcal{L}(X,X)} \leqq h_1(\sigma_1, \sigma_2)^2$$
$$\|T(\sigma_1) T(\sigma_2)^*\|_{\mathcal{L}(Y,Y)} \leqq h_2(\sigma_1, \sigma_2)^2$$

and

$$h(\sigma_1, \sigma_2) = \int h_1(\sigma_1, \sigma) h_2(\sigma, \sigma_2) d\sigma$$

is the kernel of a bounded operator on $L^2(E, d\sigma)$ with norm N^2, then

$$\left\| \int_F T(\sigma) d\sigma \right\|_{\mathcal{L}(X,Y)} \leqq N$$

where F is any subset of finite measure of E.

Proof. See Calderón-Vaillancourt [1].

Theorem 2 (Marcinkiewicz). *Let X, Y be Banach spaces, $1 < q \leqq \infty$, and let T be a sub-additive mapping from $L^1(\boldsymbol{R}^n; X) + L^q(\boldsymbol{R}^n; X)$ into the space of Y-valued strongly measurable functions on $\boldsymbol{R}^n$. Assume that for all $\lambda > 0$*

$$m\{x \mid \|Tf(x)\|_Y > \lambda\} \leqq C_1 \lambda^{-1} \|f\|_{L^1(\boldsymbol{R}^n; X)},$$
$$m\{x \mid \|Tf(x)\|_Y > \lambda\} \leqq \{C_2 \lambda^{-1} \|f\|_{L^q(\boldsymbol{R}^n; X)}\}^q$$

(when $q = \infty$ we assume that $\|Tf\|_{L^\infty(\boldsymbol{R}^n; Y)} \leqq C_2 \|f\|_{L^\infty(\boldsymbol{R}^n; X)}$). Then for all $1 < p < q$ we have

$$\|Tf\|_{L^p(\boldsymbol{R}^n; Y)} \leqq C_p \|f\|_{L^p(\boldsymbol{R}^n; X)}$$

where C_p depends only on C_1, C_2, p and q.

For the proof of the theorem see, for example, E. M. Stein [5] p. 21.

Theorem 3. *Let X, Y be Hilbert spaces, $1 < p < \infty$, and let $K(x, z, y)$ be an $\mathcal{L}(X, Y)$-valued function which satisfies the following properties:*

(I) $K \in \mathcal{C}^\infty(\boldsymbol{R}^n \times (\boldsymbol{R}^n - \{0\}) \times \boldsymbol{R}^n; \mathcal{L}(X, Y))$,

(II) *There exists $p(x, \xi, y)$ such that for $f \in \mathcal{S}(\boldsymbol{R}^n; X)$, $g \in \mathcal{S}(\boldsymbol{R}^n; Y)$*

$\int K(x, x-y, y)f(y)dy=(2\pi)^{-n}\iint p(x, \xi, y)f(y)e^{i(x-y)\xi}dyd\xi$, $\int K(x, x-y, y)^* g(x)dx=(2\pi)^{-n}\iint p(x, \xi, y)^* g(x)e^{i(x-y)\xi}dxd\xi$;

(III) *Its Fourier transform* $p(x, \xi, y)$ *with respect to* z *belongs to* $S^{\mu}_{\rho,\delta,\varepsilon}(\boldsymbol{R}^{3n}; \mathcal{L}(X, Y))$, $p(x, \xi, y)^*$ *belongs to* $S^{\mu^*}_{\rho^*,\delta^*,\varepsilon^*}(\boldsymbol{R}^{3n}; \mathcal{L}(Y, X))$, *and* $(\mu, \rho, \delta, \varepsilon)$, $(\mu^*, \rho^*, \delta^*, \varepsilon^*)$ *satisfy the conditions stated in Theorem* 1;

(IV) *For any* $|\beta|+|\gamma|=1$, $|\alpha|+|\gamma|=1$,

$$\| |z|^{n+1}D^{\beta}_{y}D^{\gamma}_{z}K(x, z, y)\|_{\mathcal{L}(X,Y)}\leqq C_{\beta\gamma}<\infty,$$
$$\| |z|^{n+1}D^{\alpha}_{x}D^{\gamma}_{z}K(x, z, y)^*\|_{\mathcal{L}(Y,X)}\leqq C_{\alpha\gamma}<\infty.$$

Then for all $f\in L^p(\boldsymbol{R}^n; X)$

$$Tf(x)=\int K(x, x-y, y)f(y)dy \tag{3}$$

is convergent in $L^p(\boldsymbol{R}^n; Y)$ *and*

$$\|Tf\|_{L^p(\boldsymbol{R}^n;Y)}\leqq C_p\|f\|_{L^p(\boldsymbol{R}^n;X)}. \tag{4}$$

Proof. (i) By Theorem 1 we obtain the conclusion for the case $p=2$.

(ii) We shall prove that for $f\in L^1(\boldsymbol{R}^n: X)$

$$m\{x\,|\,\|Tf(x)\|_Y>\lambda\}\leqq C_1\lambda^{-1}\|f\|_{L^1(\boldsymbol{R}^n;X)}, \tag{5}$$

where C_1 depends only on n, C_2, $C_{\beta\gamma}$, $(|\beta|+|\gamma|=1)$.

From Calderón-Zygumund's theorem it follows that (cf. E. M. Stein [5]) for $f\in L^1(\boldsymbol{R}^n; X)$, $\lambda>0$, there exists a decomposition of $\boldsymbol{R}^n$ so that $\boldsymbol{R}^n=F\cup\Omega$, $F\cap\Omega=\emptyset$, $\|f(x)\|_X\leqq\lambda$ almost everywhere on F, Ω is the union of cubes $\Omega=\bigcup_k Q_k$, whose interiors are disjoint, and so that for each Q_k

$$\lambda m(Q_k)\leqq\|f\|_{L^1(Q_k;X)}\leqq 2^n\lambda m(Q_k).$$

Let

$$f_0(x)=\begin{cases} f(x) & \text{for } x\in F,\\ \dfrac{1}{m(Q_j)}\displaystyle\int_{Q_j}f(y)dy & \text{for } x\in Q^0_j,\end{cases}$$

($Q^0_j=$the interior of Q_j), and let $g(x)=f(x)-f_0(x)$.

Then from the inequality

$$\|f_0(x)\|^2_{L^2(\boldsymbol{R}^n;X)}\leqq\lambda(1+2^{2n})\|f\|_{L^1(\boldsymbol{R}^n;X)},$$

we obtain

$$m\{x\,|\,\|Tf_0(x)\|_Y>\lambda\}\leqq C_2(1+2^{2n})\lambda^{-1}\|f\|_{L^1(\boldsymbol{R}^n;X)}. \tag{6}$$

Let x^k be the center of Q_k, $2b_k$ the length of the side of Q_k, $r_k=\sqrt{n}\,b_k$, and let us write $B_k=\{x\,|\,|x-x^k|\leqq 2r_k\}$, $D'=\bigcup_k B_k$, $D=\boldsymbol{R}^n\setminus D'$,

$$g_k(x)=\begin{cases} g(x) & \text{for } x\in Q^0_k,\\ 0 & \text{otherwise.}\end{cases}$$

Since the integral of g_k is equal to zero, it follows that

$$\begin{aligned} Tg_k(x)&=\int K(x, x-y, y)g_k(y)dy\\ &=\int\{K(x, x-y, y)-K(x, x-x^k, x^k)\}g_k(y)dy.\end{aligned}$$

Notice that

$$K(x, x-y, y)-K(x, x-x^k, x^k)$$
$$=\int_0^1 \{\mathrm{grad}_z K(x, x-y(t), y(t))-\mathrm{grad}_y K(x, x-y(t), y(t))\}(x^k-y)dt,$$

where $y(t)=y+t(x^k-y)$, so that

$$\|K(x, x-y, y)-K(x, x-x^k, x^k)\|_{\mathcal{L}(X,Y)}\leqq C'r_k|x-x^k|^{-n-1},$$

for $x \notin B_k, y \in Q_k$, since $|x^k-y|, |x^k-y(t)|\leqq r_k$ and

$$|x-y(t)|\geqq|x-x^k|-|x^k-y(t)|\geqq|x-x^k|-r_k\geqq\frac{1}{2}(x-x^k).$$

Thus we have

$$\int_D \|Tg_k(x)\|_Y dx\leqq C'\sqrt{n}\, b_k \int_D |x-x^k|^{-n-1}dx\|g_k\|_{L^1(\boldsymbol{R}^n;X)},$$
$$\leqq C'na_n\|g_k\|_{L^1(\boldsymbol{R}^n;X)}.$$

Therefore

$$\int_D \|Tg(x)\|_Y dx\leqq C''\int\sum_{j=1}^{\infty}\|g_j(x)\|_X dx=C''\int\|g(x)\|_X dx,$$
$$\leqq C''\{\|f\|_{L^1(\boldsymbol{R}^n;X)}+2^n\lambda m(\Omega)\}\leqq C''(1+2^n)\|f\|_{L^1(\boldsymbol{R}^n;X)},$$

and from this it follows that

$$m(D\cap\{x\,|\,\|Tg(x)\|_Y>\lambda\})\leqq C''\lambda^{-1}\|f\|_{L^1(\boldsymbol{R}^n;X)}.$$

Since $m(D')\leqq a_n n^{n/2}m(\Omega)$, it follows that $m\{x\,|\,\|Tg(x)\|_Y>\lambda\}\leqq C_1\lambda^{-1}\|f\|_{L^1(\boldsymbol{R}^n;X)}$, which, combining with the inequality (6), gives the estimate (5), since

$$m\{x\,|\,\|Tf(x)\|_Y>2\lambda\}\leqq m\{x\,|\,\|Tf_0(x)\|_Y>\lambda\}+m\{x\,|\,\|Tg(x)\|_Y>\lambda\}.$$

(iii) Case $1<p<2$. T is well defined for $L^1(\boldsymbol{R}^n; X)+L^2(\boldsymbol{R}^n; X)$ and also linear. By the result (i), (ii) and Theorem 2 we obtain the conclusion for the case.

(iv) Case $2<p<\infty$. Let p' denote the conjugate exponent of p. From Fubini's theorem it follows that for $f\in\mathcal{S}(\boldsymbol{R}^n; X)$, $g\in\mathcal{S}(\boldsymbol{R}^n; Y)$

$$\int(Tf(x), g(x))_Y dx=\int dy\left(f(y), \int K(x, x-y, y)^*g(x)dx\right)_X,$$
$$=\int(f(y), T^*g(y))_X dy,$$

where T^* is the operator with the kernel $K(y, -z, x)^*$.
But the theorem is valid for $1<p'<2$. Consequently

$$\left|\int(Tf(x), g(x))_Y dx\right|\leqq\|f\|_{L^p(\boldsymbol{R}^n;X)}\|T^*g\|_{L^{p'}(\boldsymbol{R}^n;X)},$$
$$\leqq C_{p'}^*\|f\|_{L^p(\boldsymbol{R}^n;X)}\|g\|_{L^{p'}(\boldsymbol{R}^n;Y)}.$$

This gives (4) in view of the duality between $L^p(\boldsymbol{R}^n; X)$ and $L^{p'}(\boldsymbol{R}^n; X)$, (see Phillips [4]). Since $\mathcal{S}(\boldsymbol{R}^n; X)$ is dense in $L^p(\boldsymbol{R}^n; X)$, this completes the proof of the theorem.

Corollary. *Let X be a Hilbert space and let $K(t, x, z, y)$ be a $\mathcal{B}^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$-valued continuous function of $t\in I=\{t\,|\,0\leqq t\leqq a\}$. Assume that K has compact support $\{z\,|\,|z|\leqq b\}$ in z, and that*

$$\int K(t,x,z,y)dz=0. \tag{7}$$

Then for any $1<p<\infty$ *and*

(I) *for* $f\in L^p(\boldsymbol{R}^n;X)$ *the integral*

$$\int_0^a t^{-n-1}dt\int K(t,x,(x-y)/t,y)f(y)dy$$

is convergent in $L^p(\boldsymbol{R}^n;X)$ *and defines a bounded linear operator from* $L^p(\boldsymbol{R}^n;X)$ *into* $L^p(\boldsymbol{R}^n;X)$.

(II) *For* $f\in L^p(\boldsymbol{R}^n;X)$ *the integral*

$$t^{-n}\int K(t,x,(x-y)/t,y)f(y)dy$$

defines a bounded linear operator from

$$L^p(\boldsymbol{R}^n;X) \quad \text{into} \quad L^p(\boldsymbol{R}^n;L^2(I,t^{-1}dt;X)).$$

(III) *For* $u(t,x)\in L^p(\boldsymbol{R}^n;L^2(I,t^{-1}dt;X))$ *the integral*

$$\int_0^a t^{-n-1}dt\int K(t,x,(x-y)/t,y)u(t,y)dy$$

is convergent in $L^p(\boldsymbol{R}^n;X)$ *and defines a bounded linear operator from* $L^p(\boldsymbol{R}^n;L^2(I,t^{-1}dt);X)$ *into* $L^p(\boldsymbol{R}^n;X)$.

Proof. Setting

$$p(t,x,\xi,y)=\int K(t,x,z,y)e^{-iz\xi}dz,$$

we first observe that for any $f\in\mathcal{S}(\boldsymbol{R}^n;X)$,

$$t^{-n}\int K(t,x,(x-y)/t,y)f(y)dy=\frac{1}{(2\pi)^n}\iint p(t,x,t\xi,y)f(y)e^{i\xi(x-y)}dyd\xi.$$

Next we observe that

$$\|D_x^\alpha D_\xi^\beta p(t,x,t\xi,y)\|_{L^q(I,t^{-1}dt)}\leqq C_{\alpha\beta q}(1+|\xi|)^{-|\beta|}, \tag{8}$$

for any $1\leqq q\leqq\infty,\alpha,\beta$. In fact, from the inequality

$$|D_x^\alpha D_\xi^\beta p(t,x,\xi,y)|\leqq C_{\alpha\beta}=\sup_{t,x,y,z}|K_\beta^{(\alpha)}(t,x,z,y)|a_nb^n,$$

where $K_\beta^{(\alpha)}(t,x,z,y)=(-iz)^\beta D_x^\alpha K(t,x,z,y)$, and the inequality

$$\begin{aligned}&|D_x^\alpha D_\xi^\beta p(t,x,\xi,y)|\\ &=2^{-k}\left|\int\sum_{j=0}^k(-1)^j\binom{k}{j}K_\beta^{(\alpha)}\left(t,x,z+\frac{j\xi\pi}{|\xi|^2},y\right)e^{-iz\xi}dz\right|\leqq C_{\alpha\beta k}|\xi|^{-k},\end{aligned} \tag{9}$$

where

$$C_{\alpha\beta k}=a_n(b+k\pi)^n2^{-k}\sup_{|\gamma|=k}\sup_{t,x,z,y}|D_z^\gamma K_\beta^{(\alpha)}(t,x,z,y)|,$$

it follows

$$\|D_x^\alpha D_\xi^\beta p(t,x,t\xi,y)\|_{L^q(I,t^{-1}dt)}\leqq C_{\alpha\beta}\left(\int_0^a t^{q|\beta|-1}dt\right)^{1/q}=C_{\alpha\beta q}a^{|\beta|},$$

for $|\xi|\leqq1,|\beta|\geqq1$. And for $|\xi|\geqq1,|\beta|\geqq1$ we have, taking $k>|\beta|$,

$$\begin{aligned}&\|D_x^\alpha D_\xi^\beta p(t,x,t\xi,y)\|^q_{L^q(I,t^{-1}dt)}\\ &\leqq C^q_{\alpha\beta}\int_0^{1/|\xi|}t^{q|\beta|-1}dt+C^q_{\alpha\beta k}|\xi|^{-kq}\int_{1/|\xi|}^\infty t^{q|\beta|-kq-1}dt=C'^q_{\alpha\beta}|\xi|^{-|\beta|q}.\end{aligned}$$

Also from (9) and the inequality

$$|D_x^\alpha p(t,x,\xi,y)|\leqq\left|\int D_x^\alpha K(t,x,z,y)(e^{-i\xi z}-1)dz\right|\leqq C_\alpha|\xi|,$$

where $C_\alpha = a_n b^{n+1} \sup_{t,x,y,z} |D_x^\alpha K(t,x,z,y)|$, it follows that

$$\|D_x^\alpha p(t,x,t\xi,y)\|^q_{L^q(I,t^{-1}dt)}$$
$$\leqq C_\alpha^q \int_0^{1/|\xi|} |\xi|^q t^{q-1} dt + C_{\alpha 01}^q \int_{1/|\xi|}^\infty |\xi|^{-q} t^{-q-1} dt = C_{\alpha q}^q.$$

And hence (8) is proved. Similarly, we obtain

$$\|D_y^\alpha D_\xi^\beta p(t,x,t\xi,y)\|_{L^q(I,t^{-1}dt)} \leqq C_{\alpha\beta q}(1+|\xi|)^{-|\beta|}. \tag{10}$$

Finally,

$$\|t^{-n}|z|^{n+1} D_x^\alpha D_y^\beta K(t,x,z/t,y)\|_{L^q(I,t^{-1}dt)}$$
$$\leqq C'_{\alpha\beta}\left\{\int_{|z|/b}^\infty t^{-nq-1}dt\right\}^{1/q} |z|^{n+1} = C'_{\alpha\beta q}|z|b^n \leqq C'_{\alpha\beta q} b^{n+1} s,$$

for $|z| \leqq bs$, and for $|\alpha|=1$,

$$\|t^{-n}|z|^{n+1} D_z^\alpha K(t,x,z/t,y)\|_{L^q(I,t^{-1}dt)}$$
$$\leqq C_\alpha \left\{\int_{|z|/b}^\infty t^{-nq-q-1}dt\right\}^{1/q} |z|^{n+1} = C_{\alpha q} b^{n+1}.$$

Now in order to apply the theorem to the operators in the corollary there only remains to observe that for the operators given by

$$T_1\zeta = \int_0^a \varphi(t)\zeta t^{-1}dt \qquad \text{for } \zeta \in X,$$
$$T_2\zeta = \varphi(t)\zeta \qquad \text{for } \zeta \in X,$$
$$T_3 f = \int_0^a \varphi(t) f(t) t^{-1} dt \qquad \text{for } f \in L^2(I, t^{-1}dt; X),$$

where φ is a measurable function, their norms are majorized as

$$\|T_1\|_{\mathcal{L}(X,X)} \leqq \|\varphi\|_{L^1(I,t^{-1}dt)},$$
$$\|T_2\|_{\mathcal{L}(X;L^2(I,t^{-1}dt;X))} = \|T_3\|_{\mathcal{L}(L^2(I,t^{-1}dt;X),X)} = \|\varphi\|_{L^2(I,t^{-1}dt)}.$$

This completes the proof of the corollary.

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