53. *Theorems on the Finite-dimensionality of Cohomology Groups. III*

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The purpose of this note is to present some theorems on finite-dimensionality of cohomology groups attached to an elliptic system $\mathcal{M}$ of linear differential equations defined on an open manifold M with smooth boundary. We also give a theorem (Theorem 3), which may be regarded as a generalization of Martineau's duality theorem (Martineau [8]). The differential operators are always assumed to be of real analytic coefficients and the manifold under consideration is always assumed to be real analytic. We use the same notations as in our previous notes [4],[5] and do not repeat their definitions if there is no fear of confusions. The detailed arguments of this note shall be given somewhere else. The present writer expresses his heartiest thanks to Mr. M. Kashiwara for many valuable discussions concerning the theory of derived category.

The conditions on M and the system $\mathcal{M}$ imposed in Theorems 1 and 3 are the following:

(1) M is a relatively compact open submanifold of a (paracompact) manifold L.

(2) The boundary N of M is non-singular, i.e. a real analytic submanifold of L.

(3) The system $\mathcal{M}$ is an admissible system defined on L, i.e. there exists locally a coherent left $\mathcal{D}_L^f$-Module $\mathcal{M}^f$ such that $\mathcal{M}=\mathcal{D}_L\underset{\mathcal{D}_L^f}{\otimes}\mathcal{M}^f$.

(4) The system $\mathcal{M}$ is elliptic on L, i.e. its characteristic variety V never intersects $\sqrt{-1}S^*L$.

(5) The system $\mathcal{M}$ admits a resolution of the following form:

$$0\leftarrow\mathcal{M}\leftarrow\mathcal{D}_L^{r_0}\leftarrow\mathcal{D}_L^{r_1}\leftarrow\cdots\leftarrow\mathcal{D}_L^{r_q}\leftarrow\mathcal{D}_L^{r_{q+1}}.$$

(In Theorem 3 we assume further that it has a free resolution of length d by $\mathcal{D}_L$.)

Before stating our theorems we prepare some notations related to the boundary value problem for an elliptic system of linear differential equations developed in Kashiwara-Kawai [2], [3]. Note that the codimension of N in L is 1 in our case and that this fact makes the situation very simple.

Let X and Y be a complex neighborhood of L and N respectively. For any x in N, ϑ_x shall denote the outer normal there.

Let $G_\pm$ denote the open subset of S_N^*X defined by $\{(x,\zeta)\in S_N^*X \mid \mathrm{Re}\,\zeta = \pm\vartheta_x\}$. Then the ellipticity assumption on $\mathcal{M}$ allows us to define the following admissible system $\mathcal{N}_\pm$ of pseudo-differential equations on $S_N^*Y \cong \sqrt{-1}S^*N$ (or on its complex neighborhood) by

(6) $$\mathcal{N}_\pm = \rho_*(\mathcal{P}_{Y\to X} \underset{\mathcal{D}_X}{\otimes} \mathcal{M}|_{G_\pm}),$$

where ρ denotes the canonical projection from $S_N^*X - S_Y^*X$ to S_N^*Y. See Sato-Kawai-Kashiwara [9] Chapter II about the properties of $\mathcal{P}_{Y\to X}$, which is, by the definition, the sheaf on $P_Y^*(X\times Y) = P^*X \underset{X}{\times} Y$.

Now our main result is the following.

Theorem 1. *Assume that the system* $\mathcal{N}_+$ *is either* $(q+1)$*-convex or* $(q-1)$*-concave at any points in its real characteristic variety.* (*See Sato-Kawai-Kashiwara* [9] *Chapter III Theorem* 2.3.10 *and Remark after it about the definition of q-convexity and q-concavity of a system of pseudo-differential equations.*)

Then

(7) $$\dim_C \mathrm{Ext}^q_{\mathcal{D}}(M;\mathcal{M},\mathcal{A}) = \dim_C \mathrm{Ext}^q_{\mathcal{D}}(M;\mathcal{M},\mathcal{B}) < \infty$$

holds. Here $\mathcal{A}$ *and* $\mathcal{B}$ *denote the sheaf of real analytic functions and hyperfunctions on M respectively.*

The proof of this theorem is based on the above quoted Theorem 2.3.10 of Sato-Kawai-Kashiwara [9] Chapter III and Theorem 1 of Kashiwara-Kawai [2]. Since these theorems assumes the vanishing of $\mathrm{Ext}^{q+1}_{\mathcal{D},N}(\overline{M};\mathcal{M},\mathcal{B})$, the arguments developed in Kawai [6] § 1 succeed in this case also if we replace $\mathcal{B}(M)$ by $\mathcal{E}(M)$, the space C^∞-functions on M, which constitutes an FS-space.

Remark. This theorem is a generalization of the conjecture given by Guillemin [1] about the relation of the "positive" characteristic variety of the tangential system and the finite dimensionality of the cohomology groups, except the point that we have restricted ourselves to the consideration in real analytic category, i.e. $\mathcal{M}$ is of real analytic coefficients and that L and N are real analytic manifolds.

If $L = \boldsymbol{R}^n$ and the system $\mathcal{M}$ of linear differential equations is of constant coefficients, the existence theorem of hyperfunction solutions for such a system on a convex open set (Komatsu [7]) gives as the following vanishing theorem. (Since we assume that $\mathcal{M}$ is elliptic in Theorem 2, the theorem of Ehrenpreis and Malgrange may be referred to instead of Komatsu [7].)

Theorem 2. *Assume that* $\mathcal{M}$ *is admissible and elliptic and that it is with constant coefficients. Assume further that M is an open set* (*not necessarily relatively compact*) *with* C^ω *boundary N in* $\boldsymbol{R}^n$. *Then, using the same notations as in Theorem* 1, *we can assert that*

(8) $$\mathrm{Ext}^q_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})=\mathrm{Ext}^q_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})=0$$
if $q\geqq 1$ *and if the system* $\mathfrak{N}_+$ *is either* $(q+1)$*-convex or* $(q-1)$*-concave at any point in its real characteristic variety.*

Remark. The nature of the result of Theorem 2, i.e. the vanishing of the cohomology groups, not merely the finite-dimensionality of the cohomology groups, may suggest us the reason why the theory of Stein manifold is easy in $\boldsymbol{C}^n$.

Since the finite-dimensionality of cohomology groups of a differential complex is closely related to the closed range property of a differential operator, the arguments in Theorem 1 and Kawai [6] § 1 prove the following generalization of Martineau's duality theorem.

Theorem 3. *Assume that* M *and* $\mathcal{M}$ *satisfy the conditions* (1)～(5). *Denote by* $\mathcal{L}$ *the dual system of* $\mathcal{M}$, *i.e.* $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D})$. *Assume further that* $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{L},\Omega_X)=0$ *except for* $j=0$, *where* Ω_X *denotes the sheaf of holomorphic n-forms. Then we have the following duality* (9) *between the solutions of* $\mathcal{M}$ *on* $\overline{M}$ *and the d-th cohomology group attached to* $\mathcal{L}$ *with support in* $\overline{M}$, *if* $\mathfrak{N}_+$ *is* 2*-convex at any point in its real characteristic variety.*

(9) $$\mathrm{Ext}^0_{\mathcal{D}}(\overline{M};\mathcal{M},\mathcal{A})=(\mathrm{Ext}^d_{\mathcal{D},\overline{M}}(L;\mathcal{L},\mathcal{B}\otimes\Omega_X))'.$$
Here the left hand side is equipped with its natural structure of DFS-space, $\mathrm{Ext}^d_{\mathcal{D},\overline{M}}(L,\mathcal{L},\mathcal{B}\otimes\Omega_X)$ *may be given naturally a structure of FS-space and the above duality holds as a pairing for these structures.*

Remark 1. Under the assumptions of Theorem 3, we clearly rewrite (9) as the following:

(10) $$\mathcal{S}(\overline{M})\cong(H^d_{\overline{M}}(L,\mathcal{S}'))'$$
where $\mathcal{S}$ denotes the (real analytic) solution sheaf of $\mathcal{M}$ and $\mathcal{S}'$ denotes the (hyperfunction) solution sheaf of $\mathcal{L}$. Note that the ellipticity assumption on $\mathcal{M}$ implies that of $\mathcal{L}$. Hence it is not needed to distinguish the real analytic solutions and the hyperfunction solution of $\mathcal{L}$.

Remark 2. Martineau's duality theorem (Martineau [8]) corresponds to the case where M is a Stein open set and $\mathcal{M}$ is a Cauchy-Riemann equation.

References

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