98. *Expandability and Product Spaces*

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(Comm. by Kenjiro SHODA, M. J. A., June 12, 1973)

1. Introduction. Let $\mathfrak{m}$ be an infinite cardinal number. A topological space X is said to be $\mathfrak{m}$-*expandable* (resp. *discretely* $\mathfrak{m}$-*expandable*), if for every locally finite (resp. discrete) collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of X with $|\Lambda| \leq \mathfrak{m}$, where $|\Lambda|$ denotes the power of Λ, there exists a locally finite collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda$. A collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of subsets of a topological space is said to be *hereditarily conservative* (H.C.) if every collection $\{H_\lambda \mid \lambda \in \Lambda\}$, such that $H_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda$, is closure preserving. A topological space X is said to be *H.C.* $\mathfrak{m}$-*expandable* (resp. *discretely H.C.* $\mathfrak{m}$-*expandable*), if for every locally finite (resp. discrete) collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of X with $|\Lambda| \leq \mathfrak{m}$, there exists a hereditarily conservative collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda$. A topological space is said to be *expandable, discretely expandable, H.C. expandable* or *discretely H.C. expandable*, respectively, if it is $\mathfrak{m}$-expandable, discretely $\mathfrak{m}$-expandable, H.C. $\mathfrak{m}$-expandable or discretely H.C. $\mathfrak{m}$-expandable for every cardinal number $\mathfrak{m}$ ([1], [2]).

In [1] and [2], Krajewski and Smith showed the following:

(i) X is $\aleph_0$-expandable if and only if X is countably paracompact.

(ii) X is $\mathfrak{m}$-expandable if and only if X is discretely $\mathfrak{m}$-expandable and countably paracompact.

(iii) X is collectionwise normal if and only if X is discretely expandable and normal.

Let $T(\mathfrak{m})$ be a set whose power is $\mathfrak{m}$ and t_0 be a distinguished element of $T(\mathfrak{m})$. On $T(\mathfrak{m})$ we define a topology by the following: A subset of $T(\mathfrak{m})$ is open if and only if it dose not contain t_0 or its complement is finite. Then $T(\mathfrak{m})$ is a compact Hausdorff space. If X is a topological space, then in the product space $X \times T(\mathfrak{m})$ let $X_0 = X \times \{t_0\}$.

The main purpose of this paper is to show the following theorem which is a generalization of Martin [3, Lemma 1].

Theorem 1. *The following statements are equivalent for a topological space X.*

(a) *X is $\mathfrak{m}$-expandable.*

(b) *$X \times T(\mathfrak{m})$ is $\mathfrak{m}$-expandable.*

(c) *$X\times T(\mathfrak{m})$ is discretely $\mathfrak{m}$-expandable.*

(d) *$X\times T(\mathfrak{m})$ is H.C. $\mathfrak{m}$-expandable.*

(e) *$X\times T(\mathfrak{m})$ is discretely H.C. $\mathfrak{m}$-expandable.*

(f) *If F is a closed subset of $X\times T(\mathfrak{m})$ with $F\cap X_0=\emptyset$, then there exists an open subset G of $X\times T(\mathfrak{m})$ such that $F\subset G$ and $\overline{G}\cap X_0=\emptyset$.*

The proof will be given in § 2.

Corollary. *If $X\times T(\mathfrak{m})$ is normal, then X is $\mathfrak{m}$-expandable.*

Using Theorem 1, we can prove the following two theorems by the same argument as in [3].

Theorem 2. *Let $f: X\to Y$ be a continuous closed mapping from an $\mathfrak{m}$-expandable space X onto a topological space Y, and let i be the identity mapping on $T(\mathfrak{m})$. If $f\times i$ is a hereditarily quotient mapping, then Y is $\mathfrak{m}$-expandable.*

Theorem 3. *The image of an $\mathfrak{m}$-expandable space under a continuous, closed, bi-quotient mapping is $\mathfrak{m}$-expandable. Hence the image of an expandable space under a continuous, closed, bi-quotient mapping is expandable.*

Finally, let X be a collectionwise normal space which is not countably paracompact (cf. Rudin [4]). Then X is discretely $\mathfrak{m}$-expandable and not $\mathfrak{m}$-expandable for every infinite cardinal number $\mathfrak{m}$. Hence $X\times T(\mathfrak{m})$ is not discretely H.C. $\mathfrak{m}$-expandable by Theorem 1. Since $T(\mathfrak{m})$ is compact, the projection $p: X\times T(\mathfrak{m})\to X$ is a perfect mapping. Hence the inverse image of a discretely (H.C.) $\mathfrak{m}$-expandable space under a perfect mapping is not necessarily discretely (H.C.) $\mathfrak{m}$-expandable. Thus we have a negative answer of problem (4) of Krajewski-Smith [2, p. 450].

2. Proof of Theorem 1.

Lemma. *Let A be a subset of $X\times T(\mathfrak{m})$ with $A\cap X_0=\emptyset$, and let $A_t=\{x\in X\mid (x,t)\in A\}$ for each $t\in T(\mathfrak{m})$. Then, the collection $\mathfrak{A}=\{A_t\mid t\in T(\mathfrak{m})\}$ of X is locally finite if and only if $\overline{A}\cap X_0=\emptyset$.*

Proof. Assume that $\mathfrak{A}$ is locally finite. Then a point x of X has a neighborhood U and a subset V of $T(\mathfrak{m})$ such that $V\ni t_0$, $T(\mathfrak{m})-V$ is finite and $U\cap A_t=\emptyset$ for each $t\in V$. Obviously, $(U\times V)\cap A=\emptyset$. Since V is a neighborhood of t_0, we have $(x,t_0)\notin \overline{A}$. Hence $\overline{A}\cap X_0=\emptyset$.

Conversely, assume $\overline{A}\cap X_0=\emptyset$. Let x be a point of X. We have a neighborhood U of x and a neighborhood V of t_0 such that $(U\times V)\cap A=\emptyset$. Then $T(\mathfrak{m})-V$ is a finite subset of $T(\mathfrak{m})$ and $U\cap A_t=\emptyset$ for each $t\in V$. Hence $\mathfrak{A}$ is locally finite.

Proof of Theorem 1. (a)→(b): By [1, Corollary 3.6.2], the product space of an $\mathfrak{m}$-expandable space and a compact space is $\mathfrak{m}$- expandable.

(b)→(c), (b)→(d), (c)→(e) and (d)→(e): These are obvious.

(e)→(f): Assume that (e) holds and let F be a closed subset of $X\times T(\mathfrak{m})$ with $F\cap X_0=\emptyset$. If we put $F_t=F\cap(X\times\{t\})$ for each $t\in T(\mathfrak{m})$, then the collection $\{F_t\mid t\in T(\mathfrak{m})\}$ is discrete. By assumption, we have a hereditarily conservative collection $\{G_t\mid t\in T(\mathfrak{m})\}$ of open subsets of $X\times T(\mathfrak{m})$ such that $F_t\subset G_t$ for each $t\in T(\mathfrak{m})$. Since $F_{t_0}=\emptyset$, we may assume $G_{t_0}=\emptyset$. Let $H_t=G_t\cap(X\times\{t\})$ for each $t\in T(\mathfrak{m})$, and let $H=\cup\{H_t\mid t\in T(\mathfrak{m})\}$. Since $X\times\{t\}$ is open and closed in $X\times T(\mathfrak{m})$ for each $t\in T(\mathfrak{m})-\{t_0\}$, H_t is open and $\overline{H}_t\subset X\times\{t\}$. Obviously, H is an open subset which contains F. Since $\{G_t\mid t\in T(\mathfrak{m})\}$ is hereditarily conservative,

$$\begin{aligned}\overline{H}&=\overline{\cup\{H_t\mid t\in T(\mathfrak{m})\}}=\cup\{\overline{H}_t\mid t\in T(\mathfrak{m})\}\\&=\cup\{\overline{H}_t\mid t\in T(\mathfrak{m})-\{t_0\}\}\subset X\times(T(\mathfrak{m})-\{t_0\}).\end{aligned}$$

Hence $\overline{H}\cap X_0=\emptyset$. Thus (f) holds.

(f)→(a): Assume that (f) holds. Let $\{F_\lambda\mid\lambda\in\Lambda\}$ be a locally finite collection of subsets of X with $|\Lambda|\leq\mathfrak{m}$. Then there is an injection $i\colon\Lambda\to T(\mathfrak{m})-\{t_0\}$. For each $t\in T(\mathfrak{m})$ we define F_t by

$$F_t=\begin{cases}F_\lambda & \text{if } t=i(\lambda),\\ \emptyset & \text{if } t\notin i(\Lambda).\end{cases}$$

Let $F=\cup\{F_t\times\{t\}\mid t\in T(\mathfrak{m})\}$, then $F\cap X_0=\emptyset$. Since $\{F_t\mid t\in T(\mathfrak{m})\}$ is locally finite, by Lemma we have $\overline{F}\cap X_0=\emptyset$. By assumption, there exists an open subset G of $X\times T(\mathfrak{m})$ such that $\overline{F}\subset G$ and $\overline{G}\cap X_0=\emptyset$. Let $G_t=\{x\in X\mid(x,t)\in G\}$ for each $t\in T(\mathfrak{m})$. Then, by Lemma, $\{G_t\mid t\in T(\mathfrak{m})\}$ is locally finite. Let $G_\lambda=G_{i(\lambda)}$ for each $\lambda\in\Lambda$, then $\{G_\lambda\mid\lambda\in\Lambda\}$ is locally finite open collection and $F_\lambda\subset G_\lambda$ for each $\lambda\in\Lambda$. Hence X is $\mathfrak{m}$-expandable. Thus (a) holds and the proof is completed.

References

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