97. Note on Generalized Atomic Sets of Formulas

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(Comm. by Kenjiro SHODA, M. J. A., June 12, 1973)

In his paper [2], H. J. Keisler introduced the concept of generalized atomic sets of formulas and made interesting investigations on the theory of models with generalized atomic sets. Recently, G. Grätzer posed the following problem ([1; Problem 71 in p. 299]): *Let F and G be generalized atomic sets. Under what conditions are the corresponding homomorphism and substructure concepts equivalent?* The purpose of this note is to give an answer to this problem. We shall actually find an answer to such a problem concerning generalized atomic sets in a wider sense.

§1. Preliminaries. Let L be a first order language with equality. A formula Φ of L which contains at most some of distinct variables $x_1, \cdots, x_n$ as free variables is denoted by $\Phi(x_1, \cdots, x_n)$ if the variables $x_1, \cdots, x_n$ need to be indicated. If $t_1, \cdots, t_n$ are terms of L, we denote by $\Phi[t_1, \cdots, t_n]$ the formula obtained from $\Phi(x_1, \cdots, x_n)$ by substituting all free occurrences of $x_1, \cdots, x_n$ by the terms $t_1, \cdots, t_n$ respectively. Let $\mathfrak{A}$ be a structure for L. The domain of $\mathfrak{A}$ is denoted by $D[\mathfrak{A}]$. Let $\Phi(x_1, \cdots, x_n)$ be any formula of L, and let $a_1, \cdots, a_n$ be any elements in $D[\mathfrak{A}]$. Then we write $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$, if $a_1, \cdots, a_n$ satisfy $\Phi(x_1, \cdots, x_n)$ in $\mathfrak{A}$ when the free variables $x_1, \cdots, x_n$ are assigned the values $a_1, \cdots, a_n$ respectively. If $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$ holds for any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, we say that Φ is valid in $\mathfrak{A}$, and denote it by $\mathfrak{A} \models \Phi$. If $\mathfrak{A} \models \Phi$ holds for every structure $\mathfrak{A}$ for L, we write $\models \Phi$. Two formulas Φ and Ψ are said to be equivalent if $\models \Phi \leftrightarrow \Psi$.

Let F be any set of formulas of L. For any subset $\mathcal{X}$ of the set $\{\wedge, \vee, \neg, \forall, \exists\}$, we denote by $\mathcal{X}F$ the set of all formulas that can be formed from the formulas in F by using only the connectives and quantifiers in $\mathcal{X}$. If $\mathcal{X}$ is a one-element set, e.g. $\mathcal{X} = \{\exists\}$, we use the briefer notation $\exists F$ in place of $\{\exists\}F$. We also abbreviate the sets $\{\wedge, \vee, \neg\}$ and $\{\wedge, \vee, \forall, \exists\}$ by the symbols $\mathcal{B}$ and $\mathcal{P}$ respectively. Moreover we denote by $[F]$ the set consisting of all formulas in F and a fixed identically false formula ϕ of L, and by $\mathcal{E}_L(F)$ or briefly $\mathcal{E}_L F$ the set of all formulas of L that are equivalent to some formulas in F.

A set F of formulas of L is said to be *generalized atomic*, if the following four conditions hold:

(1) If $\Phi(x_1, \cdots, x_n) \in F$ and y is a variable of L whose new

occurrences in $\Phi[y, x_2, \cdots, x_n]$ are all free, then $\Phi[y, x_2, \cdots, x_n] \in F$.

(2) If $\Phi(x_1, \cdots, x_n) \in F$, then $\Phi[e, x_2, \cdots, x_n] \in F$ for any constant symbol e of L.

(3) If Φ is a formula of L which is congruent to some formula in F, then $\Phi \in F$. (Here the term "congruent" is used in the similar meaning as in [3; p. 82].)

(4) $x = y \in F$, where x and y are distinct variables of L.

The above conditions are only the main part of the requirements of the definition of Keisler [2]. Hence every generalized atomic set in the sense of Keisler is generalized atomic (in our sense). Conversely, if F is a generalized atomic set of formulas of L, then $\mathcal{E}_L[F]$ is generalized atomic in the sense of Keisler. (For the substitution for free occurrences of variables in his definition of a generalized atomic set means substituting after renaming bound occurrences so that new occurrences do not get bound.)

Let F be a generalized atomic set of formulas of L, and let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L. A mapping h of $D[\mathfrak{A}]$ onto $D[\mathfrak{B}]$ is called an *F-homomorphism* of $\mathfrak{A}$ to $\mathfrak{B}$, ($\mathfrak{B}$ is called an F-homomorphic image of $\mathfrak{A}$ by h), if for any formula $\Phi(x_1, \cdots, x_n)$ in F and any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$ implies $(\mathfrak{B}; h(a_1), \cdots, h(a_n)) \models \Phi(x_1, \cdots, x_n)$. If $D[\mathfrak{A}]$ is a subset of $D[\mathfrak{B}]$ and for any formula $\Phi(x_1, \cdots, x_n)$ in F and any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$ if and only if $(\mathfrak{B}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$, then we say that $\mathfrak{A}$ is an *F-substructure* of $\mathfrak{B}$ or that $\mathfrak{B}$ is an *F-extension* of $\mathfrak{A}$, and denote it by $\mathfrak{A} \subseteq_F \mathfrak{B}$.

Let E be a set of constant symbols not belonging to L. Then we denote by $L(E)$ the language obtained from L by adjoining all constant symbols in E. Let F be a generalized atomic set of formulas of L. We denote by $F(E)$ the generalized atomic set in $L(E)$ which is generated by F, i.e. the least generalized atomic set in $L(E)$ cotaining F. Let $\mathcal{X}$ be any subset of the set $\{\wedge, \vee, \neg, \forall, \exists\}$. Then it is easy to see that $(\mathcal{X}F)(E) = \mathcal{X}(F(E))$. Hence both $(\mathcal{X}F)(E)$ and $\mathcal{X}(F(E))$ are simply denoted by $\mathcal{X}F(E)$. Now let $\mathfrak{A}$ be a structure for L, and φ a mapping of E into $D[\mathfrak{A}]$. Then $\mathfrak{A}$ can be expanded to a structure for $L(E)$ by interpreting $e \in E$ as $\varphi(e) \in D[\mathfrak{A}]$. Such an expanded structure is denoted by $\mathfrak{A}(\varphi)$.

Let Σ be a set of sentences of L. A structure $\mathfrak{A}$ for L is called a model of Σ if every sentence in Σ is valid in $\mathfrak{A}$. We denote by Σ^* the class of all models of Σ. If a sentence Ψ is valid in every model of Σ, we write $\Sigma \models \Psi$. If $\Sigma = \{\Phi\}$, we simply write $\Phi \models \Psi$ in place of $\{\Phi\} \models \Psi$. A class K of structures for L is said to be axiomatic if $K = \Sigma^*$ for some set Σ of sentences. For any class K of structures for L, we denote by K^* the set of all sentences of L that are valid in all structures in K.

Now let K be any class of structures for L. We denote by $\boldsymbol{S}_e(K)$ the class of all elementary substructures of structures in K. Moreover let F be any generalized atomic set of formulas of L. We denote by F-$\boldsymbol{H}(K)$ the class of all F-homomorphic images of structures in K, by F-$\boldsymbol{S}(K)$ the class of all F-substructures of structures in K, and by F-$\boldsymbol{E}(K)$ the class of all F-extensions of structures in K.

§2. **Homomorphisms.** Since, for any generalized atomic set F in L, $\mathcal{E}_L[F]$ is generalized atomic in the sense of Keisler and $\mathcal{E}_L[\mathcal{P}F]=\mathcal{E}_L(\mathcal{P}\mathcal{E}_L[F])$, the statement (a) of Corollary 3.2 in the paper [2] is as follows:

(#) Let F be a generalized atomic set of formulas of L. If K is an axiomatic class of structures for L, then

$$\boldsymbol{S}_e(\mathcal{E}_L[F]\text{-}\boldsymbol{H}(K))=(K^*\cap\mathcal{E}_L[\mathcal{P}F])^*.$$

Using this result, we shall prove the following:

Lemma 1. *Let F be a generalized atomic set of formulas of L. A sentence Φ of L is preserved under the formation of F-homomorphic images if and only if Φ is in $\mathcal{E}_L[\mathcal{P}F]$.*

Proof. Since the "if" part can be easily verified, we shall prove the "only if" part. It is obvious that every $\mathcal{E}_L[F]$-homomorphism is an F-homomorphism and vice versa. Hence it follows from (#) that

$$\boldsymbol{S}_e(F\text{-}\boldsymbol{H}(\{\Phi\}^*))=\Gamma^*,$$

where Γ is the set of all sentences Θ in $\mathcal{E}_L[\mathcal{P}F]$ such that $\Phi\models\Theta$. Since Φ is preserved under the formation of F-homomorphic images, we have

$$\boldsymbol{S}_e(F\text{-}\boldsymbol{H}(\{\Phi\}^*))\subseteq\{\Phi\}^*.$$

Hence $\Gamma^*\subseteq\{\Phi\}^*$, and hence $\Gamma\models\Phi$. By the Compactness Theorem, there exists a finite subset $\{\Theta_1,\cdots,\Theta_m\}$ of Γ such that $\{\Theta_1,\cdots,\Theta_m\}\models\Phi$. Hence $\models\Psi\rightarrow\Phi$, where $\Psi=\Theta_1\wedge\cdots\wedge\Theta_m$. Now we have that Ψ is in Γ, because Γ is closed under conjunction. Hence $\Phi\models\Psi$, and hence $\models\Phi\rightarrow\Psi$. Therefore we have that $\models\Phi\leftrightarrow\Psi$. Hence Φ is in $\mathcal{E}_L[\mathcal{P}F]$.

Next we shall prove the following:

Lemma 2. *Let F and G be generalized atomic sets of formulas of L. If every F-homomorphism is a G-homomorphism, then $G\subseteq\mathcal{E}_L[\mathcal{P}F]$.*

Proof. Let $\Phi(x_1,\cdots,x_n)$ be any formula in G, and let $E=\{e_1,\cdots,e_n\}$ be a set of constant symbols not belonging to L. Then $\Phi[e_1,\cdots,e_n]$ is in $G(E)$.

Now suppose h is any $F(E)$-homomorphism of $\mathfrak{A}$ to $\mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are structures for $L(E)$. Then h is an F-homomorphism of $\mathfrak{A}'$ to $\mathfrak{B}'$, where $\mathfrak{A}'$ and $\mathfrak{B}'$ are the restrictions to L of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Hence by the assumption of this lemma, h is a G-homomorphism of $\mathfrak{A}'$ to $\mathfrak{B}'$. Now let φ be the mapping of E into $D[\mathfrak{A}']$ such that $\mathfrak{A}'(\varphi)=\mathfrak{A}$. Then $\mathfrak{B}'(h\varphi)=\mathfrak{B}$. Moreover it is easy to see that h is a $G(E)$-homomorphism of $\mathfrak{A}'(\varphi)$ to $\mathfrak{B}'(h\varphi)$. Hence we have that

$$\mathfrak{A} \models \Phi[e_1, \cdots, e_n] \quad \text{implies} \quad \mathfrak{B} \models \Phi[e_1, \cdots, e_n].$$

From the above argument, we know that the sentence $\Phi[e_1, \cdots, e_n]$ is preserved under the formation of $F(E)$-homomorphic images. Hence by Lemma 1, $\Phi[e_1, \cdots, e_n]$ is in $\mathcal{E}_{L(E)}[\mathcal{P}F(E)]$. Since $\mathcal{E}_{L(E)}[\mathcal{P}F(E)] = \mathcal{E}_{L(E)}[(\mathcal{P}F)(E)]$,

$$\models \Phi[e_1, \cdots, e_n] \leftrightarrow \phi$$

or there exists a formula $\Theta(x_1, \cdots, x_n) \in \mathcal{P}F$ such that

$$\models \Phi[e_1, \cdots, e_n] \leftrightarrow \Theta[e_1, \cdots, e_n].$$

Hence it is easy to see that

$$\models \Phi(x_1, \cdots, x_n) \leftrightarrow \phi \quad \text{or} \quad \models \Phi(x_1, \cdots, x_n) \leftrightarrow \Theta(x_1, \cdots, x_n).$$

It follows from either case that $\Phi(x_1, \cdots, x_n)$ is in $\mathcal{E}_L[\mathcal{P}F]$. Therefore we have $G \subseteq \mathcal{E}_L[\mathcal{P}F]$.

The following theorem is an answer to the problem of Grätzer for homomorphisms.

Theorem 1. *Let F and G be generalized atomic sets of formulas of L. The concept of F-homomorphisms is equivalent to that of G-homomorphisms if and only if $\mathcal{E}_L[\mathcal{P}F] = \mathcal{E}_L[\mathcal{P}G]$.*

Proof. Since the "if" part can be easily verified, we shall prove the "only if" part. Assume that every F-homomorphism is a G-homomorphism. Then by Lemma 2, we have $G \subseteq \mathcal{E}_L[\mathcal{P}F]$. Hence $\mathcal{P}G \subseteq \mathcal{E}_L[\mathcal{P}F]$, and hence $\mathcal{E}_L[\mathcal{P}G] \subseteq \mathcal{E}_L[\mathcal{P}F]$. Similarly, if we assume that every G-homomorphism is an F-homomorphism, then we have $\mathcal{E}_L[\mathcal{P}F] \subseteq \mathcal{E}_L[\mathcal{P}G]$. Therefore, if the concept of F-homomorphisms is equivalent to that of G-homomorphisms, then we have $\mathcal{E}_L[\mathcal{P}F] = \mathcal{E}_L[\mathcal{P}G]$.

§ 3. Substructures and extensions. Corollaries 1.3b and 2.2b in the paper [2] can be stated as follows:

(##) Let F be a generalized atomic set of formulas of L, and let K be an axiomatic class of structures for L. Then

$$\boldsymbol{S}_e(\mathcal{E}_L[F]\text{-}\boldsymbol{E}(K)) = (K^* \cap \mathcal{E}_L(\exists \mathcal{B}F))^*.$$

(###) Let F and K be the same as in (##). Then

$$\mathcal{E}_L[F]\text{-}\boldsymbol{S}(K) = (K^* \cap \mathcal{E}_L(\forall \mathcal{B}F))^*.$$

By the similar method as in the proof of Lemma 1, the following two lemmas can be obtained from (##) and (###) respectively.

Lemma 3. *Let F be a generalized atomic set of formulas of L. A sentence Φ of L is preserved under the formation of F-extensions if and only if Φ is in $\mathcal{E}_L(\exists \mathcal{B}F)$.*

Lemma 4. *Let F be a generalized atomic set of formulas of L. A sentence Φ of L is preserved under the formation of F-substructures if and only if Φ is in $\mathcal{E}_L(\forall \mathcal{B}F)$.*

Now we shall prove the following:

Lemma 5. *Let F and G be generalized atomic sets of formulas of L. If every F-extension is a G-extension (or equivalently, every F-substructure is a G-substructure), then*

$$G \subseteq \mathcal{E}_L(\exists \mathcal{B}F) \quad and \quad G \subseteq \mathcal{E}_L(\forall \mathcal{B}F).$$

Proof. Let $\Phi(x_1, \cdots, x_n)$ be any formula in G, and let $E=\{e_1, \cdots, e_n\}$ be a set of constant symbols not belonging to L. Then $\Phi[e_1, \cdots, e_n]$ is in $G(E)$.

Now suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are any structures for $L(E)$ such that $\mathfrak{A} \subseteq_{F(E)} \mathfrak{B}$. Then we have $\mathfrak{A}' \subseteq_F \mathfrak{B}'$, where $\mathfrak{A}'$ and $\mathfrak{B}'$ are the restrictions to L of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Hence by the assumption of this lemma, we have $\mathfrak{A}' \subseteq_G \mathfrak{B}'$. Now let φ be the mapping of E into $D[\mathfrak{A}']$ such that $\mathfrak{A}'(\varphi)=\mathfrak{A}$. Then $\mathfrak{B}'(\varphi)=\mathfrak{B}$ by considering φ as the mapping of E into $D[\mathfrak{B}']$. Moreover it is easy to see that $\mathfrak{A}'(\varphi) \subseteq_{G(E)} \mathfrak{B}'(\varphi)$. Hence we have that

$$\mathfrak{A} \models \Phi[e_1, \cdots, e_n] \quad \text{if and only if} \quad \mathfrak{B} \models \Phi[e_1, \cdots, e_n].$$

From the above argument, we know that the sentence $\Phi[e_1, \cdots, e_n]$ is preserved under the formation of $F(E)$-extensions and $F(E)$-substructures. Hence by Lemmas 3 and 4, we have that

$$\Phi[e_1, \cdots, e_n] \in \mathcal{E}_{L(E)}(\exists \mathcal{B}F(E)) \quad \text{and} \quad \Phi[e_1, \cdots, e_n] \in \mathcal{E}_{L(E)}(\forall \mathcal{B}F(E)).$$

Since $\exists \mathcal{B}F(E)=(\exists \mathcal{B}F)(E)$ and $\forall \mathcal{B}F(E)=(\forall \mathcal{B}F)(E)$, there exist formulas $\Theta_1(x_1, \cdots, x_n) \in \exists \mathcal{B}F$ and $\Theta_2(x_1, \cdots, x_n) \in \forall \mathcal{B}F$ such that

$$\models \Phi[e_1, \cdots, e_n] \leftrightarrow \Theta_i[e_1, \cdots, e_n] \qquad \text{for } i=1,2.$$

This follows that

$$\models \Phi(x_1, \cdots, x_n) \leftrightarrow \Theta_i(x_1, \cdots, x_n) \qquad \text{for } i=1,2.$$

Hence

$$\Phi(x_1, \cdots, x_n) \in \mathcal{E}_L(\exists \mathcal{B}F) \quad \text{and} \quad \Phi(x_1, \cdots, x_n) \in \mathcal{E}_L(\forall \mathcal{B}F).$$

Therefore we have that $G \subseteq \mathcal{E}_L(\exists \mathcal{B}F)$ and $G \subseteq \mathcal{E}_L(\forall \mathcal{B}F)$.

The following theorem gives an answer to the problem of Grätzer for substructures.

Theorem 2. *Let F and G be generalized atomic sets of formulas of L. Then the following four conditions on F and G are equivalent*:

(1) *The concept of F-extensions is equivalent to that of G-extensions*;

(2) *The concept of F-substructures is equivalent to that of G-substructures*;

(3) $\mathcal{E}_L(\exists \mathcal{B}F)=\mathcal{E}_L(\exists \mathcal{B}G)$;

(4) $\mathcal{E}_L(\forall \mathcal{B}F)=\mathcal{E}_L(\forall \mathcal{B}G)$.

Proof. (1)$\Leftrightarrow$(2) and (3)$\Leftrightarrow$(4) are obvious. First we shall prove (1)$\Rightarrow$(3). Assume that every F-extension is a G-extension. Then by Lemma 5, we have

$$G \subseteq \mathcal{E}_L(\exists \mathcal{B}F) \quad \text{and} \quad G \subseteq \mathcal{E}_L(\forall \mathcal{B}F).$$

Now let Φ be any formula in G. Then $\Phi \in \mathcal{E}_L(\forall \mathcal{B}F)$. Hence $\neg\Phi \in \mathcal{E}_L(\exists \mathcal{B}F)$. Therefore we have that $\mathcal{B}G \subseteq \mathcal{E}_L(\exists \mathcal{B}F)$, because $\mathcal{E}_L(\exists \mathcal{B}F)$ is closed under conjunction and disjunction. Hence we have that $\mathcal{E}_L(\exists \mathcal{B}G) \subseteq \mathcal{E}_L(\exists \mathcal{B}F)$. Similarly, if we assume that every G-extension is an F-extension, then the converse inclusion is obtained. Therefore, if we assume that the condition (1) holds, then we have $\mathcal{E}_L(\exists\mathcal{B}F)=\mathcal{E}_L(\exists\mathcal{B}G)$, which is the assertion in (3).

Next we shall prove (3)$\Rightarrow$(1). Assume that the condition (3) holds. Then the condition (4) also holds. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any structures such that $\mathfrak{A}\subseteq_F\mathfrak{B}$. Now suppose $\Psi(x_1,\cdots,x_n)$ is any formula in G. Then by (3) and (4), we have that $\Psi(x_1,\cdots,x_n)\in\mathcal{E}_L(\exists\mathcal{B}F)$ and $\Psi(x_1,\cdots,x_n)\in\mathcal{E}_L(\forall\mathcal{B}F)$. Hence it is easy to see that, for any elements $a_1,\cdots,a_n$ in $D[\mathfrak{A}]$,

$(\mathfrak{A};a_1,\cdots,a_n)\models\Psi(x_1,\cdots,x_n)$ implies $(\mathfrak{B};a_1,\cdots,a_n)\models\Psi(x_1,\cdots,x_n)$

and

$(\mathfrak{B};a_1,\cdots,a_n)\models\Psi(x_1,\cdots,x_n)$ implies $(\mathfrak{A};a_1,\cdots,a_n)\models\Psi(x_1,\cdots,x_n)$.

Hence we have $\mathfrak{A}\subseteq_G\mathfrak{B}$. Similarly, if we assume $\mathfrak{A}\subseteq_G\mathfrak{B}$, then we have $\mathfrak{A}\subseteq_F\mathfrak{B}$. Therefore we have the assertion in (1).

§4. Supplement. Let F and G be generalized atomic sets of formulas of L. By the similar method as in §2, the following results can be obtained from Corollaries 1.3a and 2.2a in [2] respectively:

(1) *The concept of F-expansions is equivalent to that of G-expansions if and only if* $\mathcal{E}_L[\exists\wedge\vee F]=\mathcal{E}_L[\exists\wedge\vee G]$;

(2) *The concept of F-abridgements is equivalent to that of G-abridgements if and only if* $\mathcal{E}_L[\forall\wedge\vee F]=\mathcal{E}_L[\forall\wedge\vee G]$.

(For the definitions of F-expansions and F-abridgements which are generalizations of the notion of F-homomorphisms, see [2; p. 6].)

References

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[3] S. C. Kleene: Mathematical Logic. John Wiley & Sons (1967).