90. On Normal Approximate Spectrum. V

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(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1973)

1. Introduction. In our previous notes [4]–[7] and [9], we have discussed some properties of the normal approximate spectra of operators on a Hilbert space $\mathfrak{H}$.

A complex number λ is an *approximate propervalue* of an operator T on $\mathfrak{H}$ if there is a sequence $\{x_n\}$ of unit vectors in $\mathfrak{H}$ such that

$$(*)\qquad \|(T-\lambda)x_n\|\to 0 \qquad (n\to\infty).$$

$\{x_n\}$ is called a normal approximate propervectors belonging to λ. The set $\pi(T)$ of all approximate propervalues is called the *approximate spectrum* of T. If $\{x_n\}$ satisfies $(*)$ and

$$(**)\qquad \|(T-\lambda)^*x_n\|\to 0 \qquad (n\to\infty),$$

then λ is called a *normal approximate propervalue* of T and $\{x_n\}$ normal approximate propervectors belonging to λ. The set $\pi_n(T)$ of all normal approximate propervalues of T is called the *normal approximate spectrum* of T.

Bunce [2] initiated to discuss the mutual dependency among the approximate propervalues of an operator T and the characters of the unital C^*-algebra $\mathfrak{A}$ generated by T. He established, among others, the reciprocity for hyponormal operators. The reciprocity for general operators is obtained in [4] and [9]. In the present note, we shall give an alternative proof of the reciprocity basing on the Berberian representation of an operator established by Berberian [1]:

Theorem A (Berberian). *For a Hilbert space $\mathfrak{H}$, there is a Hilbert space $\mathfrak{K}$ such that*

(i) *an operator T acting on $\mathfrak{H}$ is represented by an operator T^0 acting on $\mathfrak{K}$ which satisfies*

$$(1)\qquad \pi(T)=\pi(T^0)=\sigma_p(T^0)$$

where $\sigma_p(T^0)$ is the point spectrum of T^0, and

(ii) *the Berberian representation: $T\to T^0$ is $*$-isomorphic and isometric.*

In the remainder of the present note, we shall give another proofs of theorems of Hildebrandt [8] and Bunce [3] also basing on the Berberian representation.

2. Reciprocity. Let $\mathfrak{A}$ be the C^*-algebra generated by an operator T and the identity. By a *character* of $\mathfrak{A}$ we mean a multiplicative linear functional of $\mathfrak{A}$. We shall show the following reciprocity theorem proved in [4] and [9]:

Theorem 1. *$\lambda \in \pi_n(T)$ if and only if there is a character ϕ of $\mathfrak{A}$ such as*

$$(2) \qquad \phi(T)=\lambda.$$

We need the following lemma:

Lemma 2. *Let λ be a normal propervalue of T and x a normal propervector of unit length belonging to λ, that is,*

$$(3) \qquad Tx=\lambda x \quad \text{and} \quad T^*x=\lambda^*x.$$

Let

$$(4) \qquad \phi(A)=(Ax|x)$$

for every $A\in\mathfrak{A}$. Then ϕ is a character of $\mathfrak{A}$.

We shall give three proofs of the lemma.

First proof. By (3), we have (2) and

$$(5) \qquad \phi(T^*)=\lambda^*=\phi(T)^*,$$

so that we have

$$\phi(T^{*m}T^n)=(T^{*m}T^nx|x)=(T^nx|T^mx)=\lambda^n\lambda^{*m}=\phi(T)^n\phi(T^*)^m$$

for $m,n=0,1,2,\cdots$. Similarly, we have

$$\phi(T^mT^{*n})=\phi(T)^m\phi(T^*)^n.$$

Hence ϕ is multiplicative on $\mathfrak{P}$ which is the algebra of all polynomials of the form $p(\lambda,\lambda^*)$. Since $\mathfrak{P}$ is dense in $\mathfrak{A}$ and ϕ is bounded, ϕ is multiplicative on $\mathfrak{A}$.

Second proof. Let

$$\ker\phi=\{A\in\mathfrak{A}\,;\,\phi(A)=0\}.$$

Since, by (3), we have

$$(ATx|x)=\lambda(Ax|x)=0,$$
$$(TAx|x)=(Ax|T^*x)=\lambda(Ax|x)=0,$$

for every $A\in\ker\phi$, and since $\ker\phi$ is self-adjoint, $\ker\phi$ is an ideal of $\mathfrak{A}$. Therefore, ϕ is a character of $\mathfrak{A}$ since $\ker\phi$ is a maximal ideal of $\mathfrak{A}$.

Third proof (K. Tamaki). The novelty of the proof is its elementary character. It is sufficient to show that

$$(6) \qquad \phi(AT)=\phi(A)\phi(T) \quad \text{and} \quad \phi(TA)=\phi(T)\phi(A)$$

for every $A\in\mathfrak{A}$. Since (2) and $\phi(T^*T)=|\lambda|^2$, we have

$$\begin{aligned}\phi((T-\lambda)^*(T-\lambda))&=\phi(T^*T-\lambda^*T-\lambda T^*+|\lambda|^2)\\&=\phi(T^*T)-\lambda^*\phi(T)-\lambda\phi(T^*)+|\lambda|^2\\&=0.\end{aligned}$$

By the Schwarz inequality, we have

$$\begin{aligned}|\phi(AT)-\phi(A)\phi(T)|^2&=|\phi((T-\lambda)A)|^2\\&\leqq\phi((T-\lambda)^*(T-\lambda))\phi(AA^*)=0.\end{aligned}$$

Similarly, $\phi(TT^*)=|\lambda|^2$ and $\phi((T-\lambda)(T-\lambda)^*)=0$ imply

$$|\phi(TA)-\phi(T)\phi(A)|^2=0.$$

Hence, (6) is proved.

Proof of Theorem 1. Suppose $\lambda \in \pi_n(T)$. Then λ is a normal propervalue of T^0 by Theorem A. Let z be a normalized normal propervector of T^0 belonging to λ. Put

$$\phi^0(A^0) = (A^0 z \mid z)$$

for every $A^0 \in \mathfrak{A}^0$, where $\mathfrak{A}^0$ is the C^*-algebra generated by T^0 and the identity. Then ϕ^0 is a character of $\mathfrak{A}^0$ by Lemma 2. Since $\mathfrak{A}^0$ is isometrically isomorphic with $\mathfrak{A}$ by Theorem A, $\phi(A) = \phi^0(A^0)$ gives a character ϕ of $\mathfrak{A}$ with $\phi(T) = \lambda$.

The remainder half of the proof is same as that of [4; Theorem 1]. Suppose that $\lambda \notin \pi_n(T)$. Then, by [4; Lemma 1], there is $\varepsilon > 0$ such as

$$(T-\lambda)^*(T-\lambda) + (T-\lambda)(T-\lambda)^* \geqq \varepsilon. \tag{7}$$

Since ϕ is a character satisfying (2), we have

$$\begin{aligned} \varepsilon &\leqq \phi((T-\lambda)^*(T-\lambda) + (T-\lambda)(T-\lambda)^*) \\ &= \phi(T-\lambda)^*\phi(T-\lambda) + \phi(T-\lambda)\phi(T-\lambda)^* \\ &= 0, \end{aligned}$$

which is a contradiction.

3. Boundary spectra. In [5], we have proved the following theorem which is stated without proof in [8; Satz 2(ii)]. Here we shall give an alternative proof using the Berberian representation:

Theorem 3 (Hildebrandt). *If $\lambda \in \partial W(T) \cap \pi(T)$, then $\lambda \in \pi_n(T)$, where ∂S is the boundary of S.*

Proof. The hypothesis implies that $\lambda \in \partial W(T^0) \cap \sigma_p(T)$, so that λ is a normal propervalue by a theorem of Hildebrandt [8; Satz 2(i)]. The remainder of the proof runs along the line of the proof of [6; Theorem 1].

4. Joint approximate spectrum. The notion of the joint approximate spectra of operators is introduced by Bunce [3]. Following after a reformulation of Nakamoto and Nakamura [10], we shall say a set of complex numbers $\lambda_1, \cdots, \lambda_n$ is a *joint approximate propervalue* of operators $T_1, \cdots, T_n$ if there is no $\varepsilon > 0$ such as

$$\sum_{i=1}^{n} (T_i - \lambda_i)^*(T_i - \lambda_i) \geqq \varepsilon, \tag{8}$$

which is equivalent to state that there is a sequence $\{x_k\}$ of unit vectors such as

$$\|(T_i - \lambda_i)x_k\| \to 0 \qquad (i = 1, 2, \cdots, n), \tag{9}$$

as $k \to \infty$, cf. [4; §3]. The set $\pi(T_1, \cdots, T_n)$ of all joint approximate propervalues is called the *joint approximate spectrum* of operators $T_1, \cdots, T_n$. By the definition, clearly we have

$$\pi(T_1, \cdots, T_n) \subset \pi(T_1) \times \cdots \times \pi(T_n). \tag{10}$$

A main result of [3] is the following existence theorem:

Theorem 4 (Bunce). *For an abelian family of operators $T_1, \cdots, T_n$, the joint approximate spectrum is a nonvoid compact set.*

In [3; Propositions 1–2], Bunce proved the theorem through an inductive argument based on the following theorem:

Theorem 5 (Bunce). *Let $T_1, \cdots, T_n$ be commuting operators. If $(\lambda_1, \cdots, \lambda_n) \in \pi(T_1, \cdots, T_{n-1})$, then there is $\lambda_n \in \pi(T_n)$ such that $(\lambda_1, \cdots, \lambda_n) \in \pi(T_1, \cdots, T_n)$.*

Proof. In the construction of Berberian [1], $\mathfrak{K}$ includes all bounded sequence of $\mathfrak{H}$ as elements, so that $z=\{x_k\} \in \mathfrak{K}$ and

$$(T_i^0-\lambda_i)z=0 \qquad (i=1,2,\cdots,n-1), \tag{11}$$

by the hypothesis, if $\{x_k\}$ satisfies (9). Define

$$\mathfrak{M}=\{z \in \mathfrak{K}\,;\, T_i^0 z=\lambda_i z (i=1,2,\cdots,n-1)\}.$$

Then, by (11), $\mathfrak{M} \neq 0$ is a (closed) subspace of $\mathfrak{K}$ which is invariant under $T_1, \cdots, T_n$ since they are commuting operators.

If there is no $(\lambda_1, \cdots, \lambda_n) \in \pi(T_1, \cdots, T_n)$, then there is an $\varepsilon>0$ which satisfies (8), so that we have

$$\sum_{i=1}^{n} (T_i^0-\lambda_i)^*(T_i^0-\lambda_i) \geqq \varepsilon,$$

by Theorem A. Hence we have

$$\varepsilon \leqq \sum_{i=1}^{n} \|(T_i^0-\lambda_i)y\|^2=\|(T_n^0-\lambda_n)y\|^2 \tag{12}$$

for every $y \in \mathfrak{M}$ with $\|y\|=1$. From (12), we can deduce that $\lambda_n \notin \pi(T_n^0|\mathfrak{M})$ where $T_n^0|\mathfrak{M}$ is the restriction of T_n^0 on $\mathfrak{M}$. Since λ_n was arbitrary, we can finally deduce that $\pi(T_n^0|\mathfrak{M})$ is empty which is a contradiction.

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