109. On the Global Existence of Real Analytic Solutions of Systems of Linear Differential Equations with Constant Coefficients

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In this note we shall give a necessary and sufficient condition for the global existence of real analytic solutions of systems of linear differential equations with constant coefficients. Recently L. Hörmander [1] has given a necessary and sufficient condition for single equations. Our result is a direct extension of Hörmander's.

1. Statements of the problem and the theorem. Let A be the ring of linear partial differential operators with constant coefficients in C^n. We may consider $A=C[\zeta_1,\cdots\zeta_n]$. Let M be an A module of finite type. Then it has a representation

(1.1) $$0\longleftarrow M\longleftarrow A^t\overset{P(\zeta)}{\longleftarrow}A^s$$

where $P(\zeta)$ is a $t\times s$ matrix with elements in A, and we can consider the system of equations with constant coefficients ${}^tP(D)$ where $D=(D_1,\cdots,D_n)$ and $D_i=-\sqrt{-1}\partial/\partial x_i$. But such a representation is not unique and it is not ${}^tP$ but M that has an intrinsic meaning. Therefore we call M a system. (See V.P. Palamodov [2], M. Kashiwara [3], and M. Sato, T. Kawai and M. Kashiwara [4].)

Now let Ω be a convex domain in R^n and $\mathcal{A}(\Omega)$ be the set of real analytic functions in Ω. $\mathrm{Ext}^1_A(M,\mathcal{A}(\Omega))$ gives the obstruction of the global existence of real analytic solutions of inhomogeneous system ${}^tP(D)u=f$ where f satisfies compatibility conditions ${}^tQ(D)f=0$. Our problem is when

(1.2) $$\mathrm{Ext}^1_A(M,\mathcal{A}(\Omega))=0$$

is valid. Note that $\mathrm{Ext}^1_A(M,\mathcal{A}(\Omega))$ is independent of the choice of the representation (1.1).

Before stating our theorem let us recall some notions in commutative algebra. (See J. P. Serre [5] and Palamodov [2].) Let $0=M_1\cap\cdots\cap M_l$ be a primary decomposition of the submodule 0 in M. Ass (M) is the set of associated prime ideals of M, that is, the set of radicals $\boldsymbol{p}_i=r_M(M_i)=\{a\in A\,;\,\exists q\in Z_+\ \ a^qM\subset M_i\}$ $(i=1,\cdots,l)$. $V(M)=\{V_1,\cdots,V_l\}$ is the set of characteristic varieties, that is, the set of irreducible algebraic varieties associated to ideals in Ass (M).

Now we introduce the notion of components at infinity of characteristic varieties. (Cf. Sato, Kawai and Kashiwara [4] which introduced the notion of supports of systems.) Let V be an irreducible algebraic variety in C^n and $\boldsymbol{p}$ its defining prime ideal. Let $\boldsymbol{p}^\infty$ be the homogeneous ideal generated by $\{P_m \in A\,;\, P_m$ is the principal part of some P in $\boldsymbol{p}\}$ and $V^\infty = V(\boldsymbol{p}^\infty)$. We call V_i^∞ $(i=1,\cdots,l)$ the components at infinity of characteristic varieties of M. We write $V^\infty(M)$ for $\{V_1^\infty,\cdots,V_l^\infty\}$.

Now we can state conditions for (1.2) following the genius of Hörmander. Let K, K' be compact convex sets in Ω and $\delta>0$. We say that the Phragmén-Lindelöf-Hörmander principle is valid for V^∞ if every plurisubharmonic function $\varphi(\zeta)$ in C^n with

$$\varphi(\zeta)\leqq H_K(\operatorname{Im}\zeta)+\delta|\zeta| \qquad \text{for } \zeta\in C^n,$$
$$\varphi(\xi)\leqq 0 \qquad \text{for } \xi\in V^\infty\cap R^n$$

also has the bound

$$\varphi(\zeta)\leqq H_{K'}(\operatorname{Im}\zeta) \qquad \text{if } \zeta\in V^\infty\cap C^n.$$

Here $H_K(\eta)=\sup_{x\in k}\langle x,\eta\rangle$. Our theorem is as follows.

Theorem. *Let Ω be an open convex set in R^n.* $\operatorname{Ext}_A^1(M,\mathcal{A}(\Omega))=0$ *if and only if for every compact set $K\subset\Omega$ there exist another compact set $K'\subset\Omega$ and $\delta>0$ so that the Phragmén-Lindelöf-Hörmander principle is valid for any $V_i^\infty\in V^\infty(M)$.*

Corollary. $\operatorname{Ext}_A^i(M,\mathcal{A}(\Omega))=0$ $(i\geqq 2)$ *is valid for an arbitrary system.*

2. Outline of the proof. To work the analytic machinery of Hörmander we need two considerations. One is geometric and about the relation between V_i and V_i^∞ and the other is algebraic and about reducing the problem to the case where M is coprimary. (See Serre [5].)

Proposition 1. *Let V be a k-dimensional irreducible algebraic variety in C^n and $V_\nu=\{\zeta\in C^n\,;\,\nu\zeta\in V\}$. Then $\lim_{\nu\to\infty}V_\nu=V^\infty$ and the multiplicity of convergence is constant on each irreducible component of V^∞.*

To explain the meaning of convergence and to prove Proposition 1 we recall "Einbettungssatz" of R. Remmert and K. Stein [6]. A slight modification of the proof of "Einbettungssatz" shows

Proposition 2. *After a suitable linear coordinate transformation if necessary, there exist polynomials $P_m^{(l)}(\zeta_1,\cdots,\zeta_k\,;\,\zeta_l)$ $(l=k+1,\cdots,n)$ with the following properties.*

(a) $\deg P^{(l)}=m_l$ and $P_{m_l}^{(l)}(0,\cdots,0\,;\,1)\neq 0$.

(b) $P^{(l)}$ has no multiple factors.

(c) Let $V^*=\{\zeta\in C^n\,;\,P^{(l)}(\zeta)=0 \quad l=k+1,\cdots,n\}$. Then V is identical with some irreducible component of V^*.

Now let $\Delta=\Delta_{k+1}\cup\cdots\cup\Delta_n$ where Δ_l is the zeros of the discriminant of $P^{(l)}$. For $(\zeta_1,\cdots,\zeta_k)\notin\Delta$ and large ν there exist m_l distinct roots

$\zeta_l^{(\mu)}(\zeta_1, \cdots, \zeta_k; \nu)$ $(\mu=1, \cdots, m_l)$ of the equation $P^{(l)}(\nu\zeta_1, \cdots \nu\zeta_k; \nu\zeta_l^{(\mu)}(\nu))=0$. Then $(\zeta_1, \cdots, \zeta_k, \zeta_{k+1}^{(\mu_{k+1})}(\nu), \cdots, \zeta_n^{(\mu_n)}(\nu))$ is in V_ν^*, and $\lim_{\nu\to\infty} V_\nu^* = V^{*\infty} = \{\zeta \in C^n ; P_m^{(l)}(\zeta)=0 \quad l=k+1, \cdots, n\}$ and the multiplicity of convergence is constant on each irreducible component of $V^{*\infty}$. It is easy to see that $\lim_{\nu\to\infty} V_\nu = V^\infty$ as point sets, hence Proposition 1 follows.

Once we hae Proposition 1, we can prove the sufficiency of Theorem following Hörmander's argument.

Let

$$0 \longleftarrow M \longleftarrow A^t \overset{P}{\longleftarrow} A^{s_0} \overset{Q_1}{\longleftarrow} A^{s_1} \overset{Q_2}{\longleftarrow} A^{s_2} \longleftarrow \cdots$$

be a free resolution of M. Then

$$\mathrm{Ext}_A^i(M, \mathcal{A}(\Omega)) = \frac{\mathrm{Ker}\,(\mathrm{Hom}_A(A^{s_{i-1}}, \mathcal{A}(\Omega))\ \mathrm{Hom}_A(A^{s_i}, \mathcal{A}(\Omega)))}{\mathrm{Im}\,(\mathrm{Hom}_A(A^{s_{i-2}}, \mathcal{A}(\Omega))\ \mathrm{Hom}_A(A^{s_{i-1}}, \mathcal{A}(\Omega)))}$$
$$= \mathrm{Ext}_A^1(A^{s_{i-2}}/Q_{i-1}A^{s_{i-1}}, \mathcal{A}(\Omega))$$

Hence to prove Corollary it is sufficient to show

Proposition 3. *If* $A^{t_1} \overset{R}{\longleftarrow} A^{t_2} \overset{S}{\longleftarrow} A^{t_3}$ *is exact,* $r_{A^{t_2}/SA^{t_3}}(0)=0$, *that is,* $\mathrm{Ass}\,(A^{t_2}/SA^{t_3})=\{0\}$.

Now we proceed on to the necessity of Theorem. With appropriate modifications for systems we can follow Hörmander's argument, but we can prove the necessity only for V_i with maximum dimension for $i=1, \cdots, l$. Hence we need the following reduction.

Proposition 4 (Serre [5] I–17 Corollaire 3). *For any associated prime ideal* $\boldsymbol{p}$ *of* M, *there exist a coprimary submodule* $N \subset M$ *such that* $r_N(0)=\boldsymbol{p}$.

Proposition 5. *Let* N *be a submodule of* M. *Then* $\mathrm{Ext}_A^1(N, \mathcal{A}(\Omega))=0$ *if* $\mathrm{Ext}_A^1(M, \mathcal{A}(\Omega))=0$.

Proof. We have an exact sequence

$$\mathrm{Ext}_A^1(M, \mathcal{A}(\Omega)) \to \mathrm{Ext}_A^1(N, \mathcal{A}(\Omega)) \to \mathrm{Ext}_A^2(M/N, \mathcal{A}(\Omega)).$$

By the assumption $\mathrm{Ext}_A^1(M, \mathcal{A}(\Omega))=0$ and by Corollary $\mathrm{Ext}_A^2(M/N, \mathcal{A}(\Omega))=0$.

References

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