108. *Multiplicative Excessive Measures of Branching Processes*

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1. Given continuous time Galton-Watson processes (abbreviated as CGW), we will consider the existence of "multiplicative" excessive measures of CGW processes. The existence of such measures is of importance in the duality of branching Markov processes. As is seen in the following theorem, no multiplicative excessive measure exists in some cases.

Given $\mu \geqq 0, \{\mu^n, n=1,2,3,\cdots\}$ is said to be *multiplicative* excessive measure (abbreviated as M-excessive measure) if

(1) $$\sum_{n=1}^{\infty} \mu^n T_t(n,m) \leqq \mu^m, \quad t \geqq 0, \quad m=1,2,3,\cdots,$$

where $T_t(n,m)$ is the transition probability of CGW process. We will denote M-excessive measures as $\hat{\mu}$. As usual,

$$h(u)=\sum_{n=0}^{\infty} q_n u^n,$$

is the generating function of $\{q_n; n \geqq 0\}$.[1] Then we have

Theorem. (i) *When CGW process is critical*[2] *there exists the unique M-excessive measure* $\hat{1}=\{1,1,\cdots\}$ *if and only if* $q_0=q_2=1/2$;
(ii) *When supercritical, there exist M-excessive measures if and only if* $q_0 \leqq 1/2$. *In this case,* $0 \leqq \mu \leqq 1/2q_0$ *gives* $\hat{\mu}$;
(iii) *When subcritical, there exist M-excessive measures if and only if* $1/2 \leqq q_0 \leqq r/2$.[3] *In this case,* $1/r \leqq \mu \leqq 1/2q_0$ *gives* $\hat{\mu}$.

The theorem is a consequence of the following

Lemma. *For* $\mu > 0$, $\hat{\mu}$ *is an M-excessive measure if and only if*

(a) $$2q_0 \leqq \mu^{-1}, \text{ and}$$

(b) $$h(\mu^{-1}) \leqq \mu^{-1}.$$ [4]

Remark. It is easy to see that the condition (b) is equivalent to (b′) $q \leqq \mu^{-1} \leqq r$.

Example 1. When $q_0=0$, CGW process is supercritical, and $q=0$

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1) $q_n \geqq 0$, and $\sum_{n=0}^{\infty} q_n=1$.

2) We call CGW process is critical, supercritical, and subcritical, if $h'(1)=1$, $h'(1)>1$, and $h'(1)<1$, respectively.

3) $0 \leqq q \leqq r$ are the roots of $h(u)-u=0$.

4) To prove the lemma a characterization of excessive measures in terms of infinitesimal generator (or resolvents) provides a useful tool (cf. [2]).

and $r=1$. Therefore $\mu\geqq 1$ gives $\hat{\mu}$.

Example 2. Let $q_0+q_2=1$, $(q_0\neq 0)$. (i) When $q_0=1/2$, $\hat{1}$ is the unique M-excessive measure, (actually $\hat{1}$ is an invariant measure). (ii) When $q_0<1/2$, $1\leqq\mu\leqq 1/2q_0$ gives $\hat{\mu}$. (iii) When $q_0>1/2$, $q_2/q_0\leqq\mu\leqq 1/2q_0$ gives $\hat{\mu}$. Thus in this example, there exists one M-excessive measure at least.

Example 3. Let $q_0+q_3=1$ $(q_3\neq 0)$. (i) There exists no M-excessive measure when critical. (ii) When supercritical (i.e. $q_0<2/3$), there exists no M-excessive measure if $1/2<q_0<2/3$, while $1\leqq\mu\leqq 1/2q_0$ gives $\hat{\mu}$ for $q_0\leqq 1/2$. (iii) When subcritical (i.e. $2/3<q_0<1$), there exist M-excessive measures if and only if $q_0\geqq(1+\sqrt{5})/4$ and $1/r\leqq\mu\leqq 1/2q_0$, where $r=(-1+\sqrt{1+4q_0/q_3})/2$.

2. For an M-excessive measure $\hat{\mu}$, if exists, the transition probability H_t of $\hat{\mu}$-dual Markov process of CGW process is given by

$$H_t(n,m)=\mu^m T_t(m,n)\frac{1}{\mu^n},\qquad n,m\geqq 1.$$

We put, though somewhat confusing notation,

$$H_t^0(0,m)=\mu^m T_t(m,0),\qquad m\geqq 1.$$

Proposition 1. *For* $f=(f_1,f_2,\cdots)$ *and* $g=(g_1,g_2,\cdots)$,

(2) $H_t(f*g)(n)=\sum H_tf(n_1)H_tg(n_2)+H_tf(n)H_t^0g(0)+H_t^0f(0)H_tg(n)$,

where the summation runs over all $n_1\geqq 1$ *and* $n_2\geqq 1$ *satisfying* $n_1+n_2=n$.[5)]

Remark. When $q_0=0$, we have $H_t^0f(0)=0$. Then (2) is a characterization of "collision" Markov processes discussed in [4], [5] in connection with a probabilistic treatment of Boltzmann's equation.

Proposition 2. $\hat{\lambda}$ *is M-excessive measure of the* $\hat{\mu}$*-dual Markov process if and only if*

$$\mu q\leqq\lambda\leqq\mu r,$$

where $q\leqq r$ *are nonnegative roots of* $h(u)-u=0$.

3. Given a diffusion process on a state space S with an invariant probability measure dx, killing rate $c=1$, and branching law $(\{q_n\},\{\pi_n(x,dy)\})$, we will consider the branching diffusion process determined by the above quantities (cf. [1]). Generalizing the definition of the previous section, given a measure ν on the state space S, a measure $\hat{\nu}$ on $\boldsymbol{S}=\bigcup_{n=1}^{\infty}S^n$ is said to be *multiplicative* if the restriction $\hat{\nu}|_{S^n}$ of $\hat{\nu}$ on S^n is the n-fold product of ν.

We assume in addition there exists $\pi_n(x,\boldsymbol{y})$ which is a density kernel of $\pi_n(x,d\boldsymbol{y})$ with respect to $\widehat{dy}$. Furthermore putting

$$\bar{q}_0=\int_S q_0(x)dx,\quad \bar{q}_k=\int_S dx q_k(x)\pi_k(x,\boldsymbol{z}),\quad k=1,2,3,\cdots,$$

5) $f*g(n)=f_n+f_{n-1}g_1+\cdots+f_1g_{n-1}+g_n,\qquad n\geqq 1.$

we assume that $\{\bar{q}_k\}$ are constants and $\sum_{k=0}^{\infty} \bar{q}_k = 1$.

Theorem. *Put* $\bar{h}(u) = \sum_{k=0}^{\infty} \bar{q}_k u^k$. *For* $\mu > 0$, $\widehat{\mu dx}$ *is multiplicative excessive measure of the branching diffusion process if and only if*

(i) $\bar{q}_0 = \bar{q}_2 = 1/2$ *and* $\mu = 1$, *when* $\bar{h}'(1) = 1$;

(ii) $\bar{q}_0 \leqq 1/2$ *and* $1 \leqq \mu \leqq 1/2\bar{q}_0$, *when* $\bar{h}'(1) > 1$; *or*

(iii) $1/2 \leqq q_0 \leqq \bar{r}/2$ *and* $1/\bar{r} \leqq \mu \leqq 1/2\bar{q}_0$, *when* $\bar{h}'(1) < 1$,

where $1 \leqq \bar{r}$ *is the root of* $\bar{h}(u) - u = 0$.

Example. Let x_t be a diffusion process with reflecting barrier on a bounded connected domain of C^∞ manifold, then there exists an invariant probability measure (normalize if necessary), (cf. [3]). If we take $q_n = \text{const.}$ and $\pi_n(x, z) = 1$, then $\bar{h}(u) = \sum_{n=0}^{\infty} q_n u^n$ and the theorem can be applied.

When there exists an M-excessive measure $\hat{\mu}$, let H_t be the $\hat{\mu}$-dual Markov transition probability. We assume in the following $q_0 \equiv 0$. For a probability measure ν on S, if we put

$$u_t(B) = \hat{\nu} H_t(B) \equiv \int_S \hat{\nu}(d\boldsymbol{x}) H_t(\boldsymbol{x}, B), \qquad B \subset S,$$

then $u_t(S) \leqq 1$ and u_t satisfies

$$(3) \qquad u_t(B) = \nu Q_t^0(B) + \int_0^t dr \sum_{m=1}^{\infty} \int_{S^m} \hat{u}_r(dz) q_m^*(z) \int_S \pi_m^*(z, dy) Q_{t-r}^0(y, B),$$

where Q_t^0 is the μ-dual transition probability[6] of the diffusion process with killing (rate $c = 1$) and

$$q_m^*(z) = \int_S \mu(dy) q_m(y) \pi_m(y, z),$$

$$\pi_m^*(z, dy) = \frac{1}{q_m^*(z)} \mu(dy) q_m(y) \pi_m(y, z).^{7)}$$

The integral equation (3) of measures u_t on S is a version of "Boltzmann's equation" (cf. eg. [5]).

References

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6) We are assuming that the diffusion process has an invariant measure μ.

7) $\pi_m(x, y)$ is the density kernel of $\pi_m(x, dy)$ with respect to $\hat{\mu}$.