128. *Some Relative Notions in the Theory of Hermitian Forms*

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In the 'classical' surgery theory on compact manifolds, all Hermitian forms to be considered are nonsingular [5]. However, in recent developments in surgery theory [2], [4], we have encountered a some-what curious situation, in which a homomorphism of rings $h: R \to S$ is given, and Hermitian forms to be considered are *defined over* R and *nonsingular over* S. For example, consider a homomorphism $h: \boldsymbol{Z}[t, t^{-1}] \to \boldsymbol{Z}$ defined by $h(t)=1$. Then it is proven that the 'Witt groups' of $\pm t$-Hermitian forms over $\boldsymbol{Z}[t, t^{-1}]$ which become nonsingular over $\boldsymbol{Z}$ are *isomorphic* to the higher dimensional knot cobordism groups. See [3], [4].

In this note we shall formulate (§2) some basic notions concerning the Hermitian forms of the above type, in the framework of, or as a variant of, Wall's L-theory [5] [6], and discuss some elementary properties. We also give an algebraic proof of a cancellation theorem**) which was proven in [4] by a topological method.

Conventions. We always consider rings with 1, not necessarily commutative, satisfying the condition: *The rank of a free module over the ring is well-defined.* All modules will be finitely generated right modules. Let R be a ring, V a quotient group of $K_1(R) = GL(R)/E(R)$. A basis of a free R-module is *V-equivalent* to another basis if the transformation matrix is *V-simple*, in other words, if it represents the zero element of V. A free module with a fixed V-equivalence class of bases is said to be *V-based*, and any basis in the class is called a *V-preferred basis*. We sometimes omit the prefix 'V-' if it is obvious in the context.

1. u-quadratic forms (The main reference is [5].). We fix a ring R with (additive) involution $a \mapsto \bar{a}$ such that $\overline{ab} = \bar{b}\bar{a}$, and $\bar{\bar{a}} = a$ ($\forall a, b \in R$). Note that $\bar{1} = 1$. A unit u is *admissible* if $u \in \text{Center}(R)$ and $\bar{u} = u^{-1}$. Let M be an R-module, u an admissible unit. A *u-quadratic form* (λ, μ) *on* M consists of functions $\lambda: M \times M \to R$, $\mu: M \to R/\{a - \bar{a}u \mid$

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**) Cappell-Shaneson has also given a proof [2, Lemma 1.3]. However, a property of S-isometries (in our terminology) in their proof does not seem to be so trivial as they asserted. It will be proven in the present paper, Theorem 3.

$a \in R\}$ satisfying the following five properties:

(i) λ is R-linear in the second variable,

(ii) $\lambda(x, y)=\overline{\lambda(y, x)}u$,

(iii) $\lambda(x, x)=\mu(x)+\overline{\mu(x)}u$,

(iv) $\mu(x+y)=\mu(x)+\mu(y)+\lambda(x, y)$,

(v) $\mu(xa)=\bar{a}\mu(x)a$, for all $x, y \in M$ $a \in R$.

If M is a projective module, there is a more convenient definition [6, Theorem 1], see also [1]. One should remark that λ is u-Hermitian in the sense of Bourbaki, ALGÈBRE, ch. 9, §3, n°1.

We will call the triple (M, λ, μ) a *u-quadratic module over R.* Capital letters X, Y, etc. denote u-quadratic modules. The orthogonal sum $X \perp Y$ is defined as usual.

A u-quadratic module $X=(M, \lambda, \mu)$ is said to be *nonsingular* if M is a free V-based module and the associated R-homomorphism $A\lambda: M \to \mathrm{Hom}_R(M, R)$ defined by $(A\lambda(x))(y)=\lambda(x, y)$ is a V-simple isomorphism. Our definition is clearly more restrictive than the usual one [6].

A typical example of a nonsingular u-quadratic module is a (u-) *standard plane* $(eR\oplus fR, \lambda, \mu)$ defined by $\lambda(e, f)=1$, $\lambda(f, e)=u$, $\mu(f)=\mu(e)=0$, where $eR\oplus fR$ denotes a free module of rank 2 with basis $\{e, f\}$. An orthogonal sum of copies of it is called a (u-) *kernel.*

We quote a characterization of a kernel due to Wall [5, Lemma 5.3]:

Lemma 1. *A nonsingular u-quadratic module (M, λ, μ) is a kernel if and only if M has a free V-based submodule H, with a preferred basis extending to one of M, and so defining a preferred class of bases of M/H, such that $\lambda(H\times H)=0$, $\mu(H)=0$, and the map $M/H \to \mathrm{Hom}_R(H, R)$ induced by λ is a V-simple isomorphism.*

Such a submodule is called a *subkernel.*

2. Relative notions. We throughout fix an *onto* homomorphism $h: R\to S$ of rings with involutions such that $h(\bar{a})=\overline{h(a)}$ ($\forall a \in R$). Then the image of an admissible unit is admissible, and, as usual, a u-quadratic module $X=(M, \lambda, \mu)$ over R gives rise to an $h(u)$-quadratic module $X\otimes_R S=(M\otimes_R S, \lambda', \mu')$ over S. If the induced $h(u)$-quadratic module $X\otimes_R S$ is nonsingular, X is said to be *S-nonsingular.* (A quotient group V of $K_1(S)$ is understood to be fixed.) To abbreviate the terminology, we will henceforth refer to an S-nonsingular u-quadratic module over R as an *S-nonsingular u-form.* For an S-nonsingular u-form (M, λ, μ), $M\otimes_R S$ is an S-free module by our definition of nonsingularity, but M is not necessarily R-free. A set of elements of M, $\{x_1, \cdots, x_n\}$ is called a *pre-basis* if the image $\{x_1\otimes 1, \cdots, x_n \otimes 1\}$ is a preferred basis of $M\otimes_R S$. If M itself is R-free, we call X a *free u-form.* Also it is always assumed that a basis of M is chosen so that it is a pre-basis.

An S-nonsingular u-form $X=(M, \lambda, \mu)$ is said to be *null-cobordant* if there exists a submodule $H \subset M$, not necessarily a direct summand, such that $\lambda(H \times H)=0$, $\mu(H)=0$ and H is mapped *onto* a subkernel of $X \otimes_R S$ under the canonical mapping $M \to M \otimes_R S$. Note that $X \otimes_R S$ is, then, a kernel by Lemma 1. Following Cappell and Shaneson [2] we call such a submodule H a *pre-subkernel*. (In our previous paper [4], H was called a Seifert subkernel.) X is *stably null-cobordant* if an orthogonal sum $X \perp$ (a kernel) is *null-cobordant*.

'*Witt groups*'. Let $Q_u^V(h)$ be the Grothendieck group of all isomorphism classes of S-nonsingular u-forms, and let $\mathcal{N}_u^V(h)$ be the subgroup generated by all stably null-cobordant forms. The 'Witt group' of S-nonsingular u-forms is defined by the quotient $Q_u^V(h)/\mathcal{N}_u^V(h)$. Since this generalizes the (even dimensional) Wall groups L_{2n}, we will denote it by $\mathcal{L}_u^V(h)$. Cappell-Shaneson's Γ-groups [2] and P-groups introduced in [4] are formulated as various special cases of these $\mathcal{L}$-groups.*)

A similar construction gives the 'Witt group' $\mathcal{F}\mathcal{L}_u^V(h)$ of all S-nonsingular free u-forms. There is a natural homomorphism $\rho: \mathcal{F}\mathcal{L}_u^V(h) \to \mathcal{L}_u^V(h)$.

Proposition 2. *ρ is an isomorphism.*

3. S-isometries. A convenient class of morphisms in the category of S-nonsingular u-forms is that of *S-isometries* defined as follows: Let $X=(M, \lambda, \mu)$, $Y=(N, \xi, \eta)$ be S-nonsingular u-forms. An R-homomorphism $\varphi: M \to N$ is an *S-isometry* if (i) φ preserves u-quadratic forms: $\xi(\varphi(x), \varphi(y))=\lambda(x, y)$, $\eta(\varphi(x))=\mu(x)$ for all $x, y \in M$, and (ii) $\varphi \otimes 1: M \otimes_R S \to N \otimes_R S$ is a V-simple isomorphism. (Thus $\varphi \otimes 1$ is an isometry in the usual sense.) φ is not necessarily injective nor surjective. In this section we will prove

Theorem 3. *Let $\varphi: X \to Y$ be an S-isometry. Then X is stably null-cobordant if and only if Y is stably null-cobordant.*

The 'only if' part is not difficult. The proof of the converse is based on the following 'restricted case'.

Lemma 4. *Let $\varphi: X \to Y$ be as above, and suppose that X is a free u-form. Then if Y is stably null-cobordant, so is X.*

Proof. By adding copies of standard planes, we may assume that $Y=(N, \xi, \eta)$ is actually a null-cobordant u-form with a pre-subkernel $K \subset N$. Let $\pi: N \to N/K$ be the quotient map. The proof is divided into 3 steps.

(1) *If the composite $\pi \circ \varphi$ is onto, then X is null-cobordant.* In

*) For example, let $h: \boldsymbol{Z}[t, t^{-1}] \to \boldsymbol{Z}$ be a homomorphism $h(t)=1$. Then $\Gamma_{2n}(h) = \mathcal{L}_{(-1)^n}^{\{0\}}(h)$, and $P_{2n}(h) = \mathcal{L}_{(-1)^n t}^{\{0\}}(h)$, where $\{0\}$ stands for a trivial group.

fact $\varphi^{-1}(K)$ is a pre-subkernel of X.

(2) In the case Coker $(\pi\circ\varphi)\neq 0$, proceed as follows. Consider an orthogonal sum $Y'=Y\perp$(a standard plane with basis $\{e,f\}$). Then $K'=K\oplus eR$ is a pre-subkernel of Y'. Let z_0 be an element N, and define an R-homomorphism $\varphi' : M\oplus xR\oplus yR\to N\oplus eR\oplus fR$ by setting $\varphi'|M=\varphi$, $\varphi'(x)=z_0\oplus e$, $\varphi'(y)=f$, where $M\oplus xR\oplus yR$ is a direct sum of M and a free module with basis $\{x,y\}$. Define a u-quadratic form (λ',μ') on $M\oplus xR\oplus yR$ as follows: $(\lambda',\mu')|M=(\lambda,\mu)$, $\lambda'(m,x)=\xi(\varphi(m),z_0)$ ($\forall m\in M$), $\lambda'(m,y)=0$ ($\forall m\in M$), $\mu'(x)=\eta(z_0)$, $\mu'(y)=0$, $\lambda'(x,y)=1$, and $\lambda'(y,x)=u$. Then $\varphi' : X'\to Y'$ turns out to be an S-isometry, where $X'=(M\oplus xR\oplus yR,\lambda',\mu')$. Let $\pi' : N\oplus eR\oplus fR\to N\oplus eR\oplus fR/K'$ be the quotient map. Then it is easily verified that Coker $(\pi'\circ\varphi')\cong$ Coker $(\pi\circ\varphi)/\pi(z_0)$. Therefore by taking suitable z_0, we can make Coker $(\pi\circ\varphi)$ strictly 'smaller'.

(3) *The u-quadratic module X' constructed above is isomorphic to $X\perp$(a standard plane).*

Proof. Here we use the assumption that $X=(M,\lambda,\mu)$ is a free u-form. Let $\{m_1,\cdots,m_s\}$ be a basis of M, which is also a pre-basis. Then an isomorphism $I : X\perp$(a standard plane with basis $\{e_1,f_1\}$)$\to X'$ is explicitly constructed as follows:

$$I(e_1)=x-yc,\qquad I(f_1)=y,$$

and

$$I(m_i)=m_i-y\bar{c}_i u,\qquad i=1,\cdots,s,$$

where $c\equiv\eta(z_0)$ mod $\{a-\bar{a}u\,|\,a\in R\}$ and $c_i=\lambda'(m_i,x)$.

The proof of Lemma 4 is now obvious from (1), (2) and (3).

Proof of Theorem 3. First we make a construction. Let $\{x_1,\cdots,x_n\}$ be a pre-basis of X, and let $x_1^*,\cdots,x_n^*$ be indeterminates. We define a u-quadratic form (λ^*,μ^*) on the free module $x_1^*R\oplus\cdots\oplus x_n^*R$ as follows: $\lambda^*(x_i^*,x_j^*)=\lambda(x_i,x_j)$, $\mu^*(x_i^*)=\mu(x_i)$. The u-quadratic form $(x_1^*R\oplus\cdots\oplus x_n^*R,\lambda^*,\mu^*)$ is denoted by X^*. The canonical map $\rho^* : X^*\to X$ defined by $\rho^*(x_i^*)=x_i$ is clearly an S-isometry. We call a pair (X^*,ρ^*) a *free core* of X, or, $\{x_1,\cdots,x_n\}$*-free core* of X. See [4].

Now the proof of the 'if' part of Theorem 3 goes as follows. Suppose Y is stably null-cobordant. Let (X^*,ρ^*) be a free core of X. Then since $\varphi\circ\rho^* : X^*\to Y$ is an S-isometry and X^* is a free u-form, X^* is stably null-cobordant by Lemma 4. Therefore X is stably null-cobordant by applying the 'only if' part of Theorem 3 to the S-isometry $\rho^* : X^*\to X$. This completes the proof.

4. A cancellation theorem. We continue to fix an onto homomorphism $h : R\to S$. Let $X=(M,\lambda,\mu)$, $Y=(N,\xi,\eta)$ be S-nonsingular u-forms.

Cancellation theorem. *Suppose $X\perp Y$ and Y are stably null-*

cobordant, then X is also stably null-cobordant.

We need a lemma due to Cappell-Shaneson [2].

Lemma 5. *Let $Y=(N, \xi, \eta)$ be a stably null-cobordant u-form, then there exists a diagram of S-isometries $K \xleftarrow{\varphi} Y^* \xrightarrow{\rho^*} Y$ in which K is a kernel.*

Proof. We may assume that Y is actually null-cobordant. Let $H \subset N$ be a pre-subkernel of Y. One can choose a pre-basis $\{e_1, \cdots, e_r, f_1, \cdots, f_r\} \subset N$ so that $\{e_1, \cdots, e_r\} \subset H$. Define (Y^*, ρ^*) as an $\{e_1, \cdots, e_r, f_1, \cdots, f_r\}$-free core of Y. Let $K = \perp_{i=1}^r S_i$, where $S_i = (x_i R \oplus y_i R, \lambda_i, \mu_i)$ is a standard plane. Then an S-isometry $\varphi: Y^* \to K$ is explicitly defined as follows:

$$\varphi(e_i^*) = \sum_{j=1}^r x_j \overline{\xi(e_i, f_j)},$$

and

$$\varphi(f_k^*) = y_k + x_k \bar{c}_k + \sum_{l<k} x_l \overline{\xi(f_k, f_l)},$$

where $c_k \equiv \eta(f_k) \bmod \{a - \bar{a}u \mid a \in R\}$.

Proof of cancellation theorem. By Lemma 5, we have S-isometries $K \xleftarrow{\varphi} Y^* \xrightarrow{\rho^*} Y$ in which K is a kernel. By making orthogonal sums we have a diagram of S-isometries $X \perp K \leftarrow X \perp Y^* \rightarrow X \perp Y$, but $X \perp Y$ is stably null-cobordant by the hypothesis. Therefore, by Theorem 3, $X \perp K$ is stably null-cobordant. This completes the proof.

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