157. Vanishing Theorems with Algebraic Growth and Algebraic Division Properties

Complex Analytic De Rham Cohomology. I

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The purpose of the present note is to announce certain quantitative properties of coherent sheaves and analytic varieties. Details will appear elsewhere. Results given here are originally and primarily intended for applications to differential forms on complex analytic varieties with arbitrary singularities (cf. the end of this note). Results stated here are, however, of their own interests. Our basic purpose is to discuss vanishing theorems of certain types where quantitative properties of objects considered appear. Quantitative properties examined here are as follows: (I) Asymptotic behaviors with respect to pole loci. (II) Division properties with respect to subvarieties. Our arguments will be divided into two steps: (i) The step in which only the asymptotic behavior enters. (ii) The step where both asymptotic behaviors and division properties appear.

Notational remarks. We write linear functions and monomials as L and M. A couple, denoted by $\sigma=(\sigma_1,\sigma_2)$, is a couple of positive numbers. For a set $\{\sigma^1,\cdots,\sigma^s\}$ of couples maps $\mathcal{L}:\{\sigma^1,\cdots,\sigma^s\}\to\sigma'$ and $\mathcal{M}:\{\sigma^1,\cdots,\sigma^s\}\to\varepsilon\in\boldsymbol{R}$ are said to be of *exponential-algebraic type* ((*e.a*)-*type*) if $\sigma'=\{M_1(\sigma_1^1,\cdots,\sigma_1^s)\times\exp M_2(\sigma_2^1,\cdots,\sigma_2^s), L(\sigma_2^1+\cdots+\sigma_2^s)\}$, $\varepsilon=M_1(\sigma_1^1,\cdots,\sigma_1^s)\times\exp M_2(\sigma_2^1,\cdots,\sigma_2^s)$.

(I) We start with a datum $(\Delta(r;P_0),X,D)$ of a polydisc Δ with the center P_0 of radius r in $\boldsymbol{C}^n$, a variety* $V\ni P_0$ in Δ and a divisor $D\ni P_0$ in Δ. We write irreducible decompositions of X and D at P_0 as $X_{P_0}=\bigcup_j X_{P_0j}$ and $D_{P_0}=\bigcup_j D_{P_0j}$. Assume that D contains the singular locus of X and that $X_{P_0j}\neq D_{P_0j'}$ for any pair (j,j'). Moreover, consider a coherent sheaf $\mathfrak{F}$ admitting a resolution of the following form

$$0\longrightarrow\mathcal{O}^{d_s}\longrightarrow\cdots\xrightarrow{K_2}\mathcal{O}^{d_1}\xrightarrow{K_1}\mathfrak{F}(\subset\mathcal{O}^d)\longrightarrow 0,$$

where K's are matrices whose coefficients are meromorphic functions on X with the pole $D'=D\cap X$. A point P is near P_0 if P is in a small neighborhood of P_0. For a point near P_0, the intersection $\Delta(r;P)\cap X$ is denoted by $\Delta(r;P,X)$. Moreover, for a point $Q\in\Delta(r;P,X)-D$, we

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*) A variety and a function are always complex analytic ones in this note.

mean by $N_\sigma(Q,D)$ the neighborhood of Q *in* X defined by $N_\sigma(Q,D) = \{Q' : d(Q,Q') \leqq \sigma_1 \cdot d(Q,D)^{\sigma_2}\}$. Define a (non locally finite) covering $\mathfrak{A}_\sigma(r;P,D)$ of $\Delta(r;P,X)-D$ by $\mathfrak{A}_\sigma(r;P,D)=\{N_\sigma(Q,D) : Q \in \Delta(r;P,X) - D\}$. We formulate our problem in terms of such coverings $\mathfrak{A}$. A q-cochain $\varphi \in C^q(N(\mathfrak{A}_\sigma(r;P,D)),\mathfrak{F})$, N=nerve, is of *algebraic growth* $\alpha=(\alpha_1,\alpha_2)$ if

$$|\varphi(Q)| \leqq \alpha_1 \cdot d(Q,D)^{-\alpha_2}.$$

Then our first result is as follows

Lemma 1. *A vanishing theorem with algebraic growth.*

There exists a datum $(\mathcal{L}_1, \mathcal{L}_2, M)$ *depending on* $(X, D, \mathfrak{F})$ *only such that the following is valid.*

For a cocycle $\varphi \in Z^q(N(\mathfrak{A}_\sigma(r;P,D),\mathfrak{F})$ *of algebraic growth* α, *there exists a cochain* $\varphi' \in C^{q-1}(N(\mathfrak{A}_{\sigma'}(r';P,D),\mathfrak{F})$ *of growth* (α_1', α_2') *such that*

(1) $\delta(\varphi')=\varphi$,

(2) $(\alpha_1', \alpha_2') = (\alpha_1'' \times M(r)^{-1}, \alpha_2'')$, $\quad \sigma' = \mathcal{L}_2(\sigma)$

and $r'=M(r)$ *with* $(\alpha_1'', \alpha_2'') = \mathcal{L}_1(\alpha)$.

In the equation (1) φ is regarded as an element in $C^q(N(\mathfrak{A}_{\sigma'}(r';P,D)),\mathfrak{F})$ by taking a refinement and a restriction suitably.

Remark 1. For a domain $\sum = \Delta(r;P)\times C^N$ and for a coherent sheaf $\mathfrak{F}'$ over $\sum$, a result similar to Lemma 1 is valid (by changing the distance to D by $\sum_j |x|_j$; (x_j) are coordinates of C^N).

Remark 2. That the datum $(\mathcal{L}_1, \mathcal{L}_2, M)$ in Lemma 1 is independent of points P shows that Lemma 1 is of *semi-global* nature.

Remark 3. Conomology theories with growth conditions have recently been studied by various persons for various purposes (cf. [1], [3]–[6]). Our methods depend on examinations of Cousin integrals and of combinatorial arguments. We proceed along standard methods of discussing vanishing theorems on Stein varieties and have many similarities with works cited above. However our situation as well as our statement are, to the author's knowledge, new. Our notion of '*algebraic growth*' was inspired by the notion of '*polynomial growth*' due to R. Narasimhan (His method is that of $\bar{\partial}$-estimation of L. Hörmander (see [5]).).

(II) Now we state our basic problem in this note. We consider a proper subvariety V of X and a set of analytic functions $(f)=(f_1, \cdots, f_t)$. Let $V_{P_0} = \bigcup_j V_{P_0 j}$ be the irreducible decomposition of V at P_0. We assume the following conditions: $V_{P_0 j} \neq X_{P_0 j'}$ for any pair (j,j'), $V_{P_0 j} \neq D_{P_0 j''}$ for any pair (j,j'') and $f_i \not\equiv 0$ on $X_{P_0 j}$ for any pair (i,j). Moreover we assume that V is the zero locus of (f) *on* X. Our problem is formulated in terms of (f). Let $\mathfrak{X}^{m,0}$ be the subsheaf of $\mathcal{O}_X$ defined to be $\mathcal{O}_X(f_1^m, \cdots, f_t^m)$. This sheaf $\mathfrak{X}^{m,0}$ is our basic subject. We associate sheaves $\mathfrak{X}^{m,s}$ $(s=1,\cdots,t-1)$ to $\mathfrak{X}^{m,0}$ in the following manner:

For a multi-index $I_s=(i_1,\cdots,i_s)$ define an element $\mathfrak{f}(I_s,m)\in\mathcal{O}^{\binom{t}{s-1}}(X)$ by

$$\mathfrak{f}(I_s,m)(J_{s-1})=\begin{cases}0, & \text{if } J_{s-1}\not\subset I_s,\\ (-1)^{k-1}f^m_{i_k} & \text{if } J_{s-1}=(i_1,\cdots,\hat{i}_k,\cdots,i_s)\end{cases}$$

(for a vector $\mathfrak{g}\in\mathcal{O}^{\binom{t}{s}}$, $\mathfrak{g}(J_s)$; $J_s=(j_1,\cdots,j_s)$ is J_s-component of $\mathfrak{g}$).

Using the above elements $\mathfrak{f}(I_s,m)$, define sheaf homomorphisms $K(s,m)$ $(s=1,\cdots,t):\mathcal{O}^{\binom{t}{s}}\to\mathcal{O}^{\binom{t}{s-1}}$ by

$$K(s,m)(\mathfrak{g})=\sum_{I_s}\mathfrak{g}(I_s)\cdot\mathfrak{f}(I_s,m).$$

Note that $\mathfrak{X}^{m,0}=K(1,m)\cdot\mathcal{O}^t$. Define $\mathfrak{X}^{m,s}$ $(s=1,\cdots,t-1)$ by $\mathfrak{X}^{m,s}=K(s,m)\mathcal{O}^{\binom{t}{s+1}}$. If the Jacobian condition: $\det\dfrac{\partial(f_1,\cdots,f_t)}{\partial(x_1,\cdots,x_t)}\neq 0$ holds at each point on X (where $(x_1,\cdots,x_t)$ denote coordinates on X) then the exact sequence

$$0\longrightarrow\mathcal{O}\xrightarrow{K(t,m)}\mathcal{O}^{\binom{t}{t-1}}\longrightarrow\cdots\cdots\longrightarrow\mathcal{O}^t\xrightarrow{K(1,m)}\mathfrak{X}^{m,0}\longrightarrow 0$$

holds. In general the above sequence fails. It is, however, found that associating $\mathfrak{X}^{m,s}$ with $\mathfrak{X}^{m,0}$ is meaningful in our discussion concerned with division properties: We formulate two problems in terms of sheaves $\mathfrak{X}^{m,s}$ rather than $\mathfrak{X}^{m,0}$ only: (i) A cochain $\varphi^q\in C^q(\mathfrak{A}_\sigma(r;P,D),\mathfrak{X}^{m,s})$ is *of algebraic growth* $(\alpha_1,\alpha_2,\alpha_3)$ if φ has the following property.

If $\bigcap_{l=1}^{q+1}N_\sigma(Q_l)\neq\phi$, the element $\varphi(\bigcap_{l=1}^{q+1}N_\sigma(Q_l))$ is written as

(1) $\varphi(Q)=\sum_{I_s}\mathfrak{g}^s\cdot\mathfrak{f}(I_s,m)$; $\mathfrak{g}^s\in\mathcal{O}^{\binom{t}{s}}(\bigcap_l N_\sigma(Q_l))$

with the estimation

(2) $|\mathfrak{g}^s|\leqq\alpha_1\cdot d(Q,D)^{-\alpha_2}\cdot d(Q,V)^{\alpha_3}$.

Now our first assertion is as follows.

Theorem 1. *There exists a datum $(\mathcal{L},L_i,M_i;i=1,2,3)$ depending on $(X,V,D,(f))$ only with which the following are valid.*

For a cocycle $\varphi\in Z^q(N(\mathfrak{A}_\sigma(r,P,D),\mathfrak{X}^{m,s})$ of growth $(\alpha_1,\alpha_2,\alpha_3)$ there exists a cochain $\varphi'\in C^{q-1}(N(\mathfrak{A}_{\sigma'}(r',P,D)),\mathfrak{X}^{m,s})$ of growth $(\alpha'_1,\alpha'_2,\alpha'_3)$ such that the equation

(1) $\delta(\varphi')=\varphi$,

and

(2) $r'=M_1(r)$, $\sigma'=\mathcal{L}(\sigma)$, $(\alpha'_1,\alpha'_2,\alpha'_3)=(M_2(\alpha_1,\sigma_1,r^{-1})\cdot\exp M_3(\alpha_2,\sigma_2,\alpha_3)$, $L_1(\alpha_2+\sigma_2)$, $L_2(\alpha_3))$, *so far as* $\alpha_3\geqq L_3(m)$.

In the above statement an emphasis is put on a '*middle point*' of the asymptotic behavior and the division property: '*The independenceness of the order of the pole α'_2 from the divisible index m*' is a key point. This is possible by diminishing the given α_3 to α'_3 and will be made an essential use of in our application of Theorem 1 to differential forms.

The second problem is as follows: Given an element $\mathfrak{g}^s\in\mathcal{O}^{\binom{t}{s}}$ so that $K(s,m)(\mathfrak{g}^s)=0$ $(s\geqq 1$, if $s=0$ we do not consider an algebraic condition), find $\mathfrak{g}^{s+1} \in \mathcal{O}^{\binom{t}{s+1}} : K(s+1, m)(\mathfrak{g}^{s+1}) = \mathfrak{g}^s$. Precisely let us consider a proper subvariety V' of V. Instead we do not consider the divisor D in this case. Take a point $P \in V - V'$. $N_r(P, V')$ is a neighborhood of P in X defined to be $\{Q : d(P, Q) \leqq r\}$. An element $\mathfrak{g}^s$ $(s = 0, \cdots, t-1)$ $\in \mathcal{O}^{\binom{t}{s}}(N_r(P, V'))$ is a *testifying datum with quantitative property* (b, α_3) if the following are valid.

$(\mathrm{A})_1$ $K(s, m)(\mathfrak{g}^s) = 0$ $(s \geqq 1)$,

$(\mathrm{A})_2$ $|\mathfrak{g}^s| \leqq b d(Q, V)^{\alpha_3}$, $Q \in N_r(P, V')$ $(s = 0, 1, \cdots, t-1)$.

Then our second assertion is as follows.

Theorem 2. *Weak syzygy with quantities.*

There exists a datum $(M, M_i\ (i=1,2,3),\ L_i\ (i=1,2,3,\sigma)$ depending on $(X, V, V', (f))$ only with which the following is true.

(B) *For a testifying datum $\mathfrak{g}^s$ with quantitative property (b, α_3), there exists an element $\mathfrak{g}^{s+1} \in \mathcal{O}^{\binom{t}{s+1}}(N_r(P, V'))$ so that*

$(\mathrm{B})_1$ $K(s, m+1)\mathfrak{g}^{s+1} = \mathfrak{g}^s$,

$(\mathrm{B})_2$ $\mathfrak{g}^{s+1}$ *is of quantitative property (b', α_3'),*

where $b' = M_1(r)^{-1} \cdot M_2(b) \cdot \exp M_3(\alpha_3)$, $\alpha_3' = L_1(\alpha_3)$ and $r' = M(r)$ hold so far as $\alpha_3 \geqq L_2(m)$, $r \leqq \sigma_1 \cdot d(P, V')^{\sigma_2}$.

Remark. Problems of differential forms which we consider are as follows. Detailed arguments will be given elsewhere. Here we do a sketch of our problems: Start with a datum (X, V, D, P_0) defined in previous arguments: For differential sheaves $\Omega = \Omega_X$ and $\Omega(*D') = \Omega_X(*D') =$ sheaf of meromorphic forms with the pole $D' = D \cap X$, $\hat{\Omega}$ and $\hat{\Omega}(*D')$ are *completions of Ω and $\Omega(*D')$ along V.* Our problems are spoken in terms of the above two rings $\hat{\Omega}, \hat{\Omega}(*D')$. Concerning the ring $\hat{\Omega}(*D')$ our problem is to show the isomorphism

$$(R^* i_* \boldsymbol{C})_{P_0} \cong \mathcal{H}^*(\hat{\Omega}(*D')_{P_0})\,;\ i \text{ is the inclusion};\ i : V - D \subsetneq V.$$

This is a generalization of a well known theorem of A. Grothendieck [2]. Concerning the ring $\hat{\Omega}$ we ask, under the assumption of $X =$ smooth variety, the exact sequence: $0 \to \hat{\Omega}^0 \xrightarrow{d} \hat{\Omega}^1 \to \cdots$, and a *division property* of the integration of differential forms. Precise meaning of the above problems will be discussed, the author plans, in another announcements. Roughly Theorem 1 and Lemma 1 are analytic keys to our problems on differential forms.

References

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