6. *On the Zeros of Heck's L-Functions*

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1. Introduction. In this note we shall derive a density estimate near $\sigma=1$ of all L-functions with character defined mod $\tilde{\mathfrak{f}}$, where $\mathfrak{f}$ is an integral ideal of K, which is the extension of the rational field of degree n. Of course, such result has been already obtained in Fogels [1], and Gallagher [2] has extended his result to a large sieve type in the case $K=\boldsymbol{Q}$. He also says that Fogels' proof may be simplified by using his method in combination with the Brun-Titchmarsh inequality and Turan's power-sum method. His assertion is true even for the case of an arbitrary algebraic number field K. It is the main purpose of this note to offer to give its proof with some explicit constants. Our main tools are the same as what Gallagher has mentioned. But some of the theorems, which are well-known, are re-proved with some explicit constants. The proof will be a little more complicated than Gallagher's because we intend to get as small constants as possible. The author wishes to express his thanks to Prof. T. Tatuzawa, who encouraged him and gave him fruitful suggestions during preparing this note.

2. Notation and terminology. Let K be an algebraic number field of degree n with r_1 real conjugates and $2r_2$ complex conjugates, $\mathfrak{f}$ an integral ideal of K and $\tilde{\mathfrak{f}}$ denote a product of $\mathfrak{f}$ and q infinite prime spots of K $(0\leqq q\leqq r_1)$. Furthermore χ is a character defined mod $\tilde{\mathfrak{f}}$ and $\sum_{\text{mod }\tilde{\mathfrak{f}}}$ means the sum taken over all χ defined mod $\tilde{\mathfrak{f}}$. Functions $\mu(\mathfrak{a})$, $\varphi(\mathfrak{a})$, $\Lambda(\mathfrak{a})$ and $\psi_K(x)$ are generalizations of the Möbius, Euler, von Mangoldt and Chebychev function to K, respectively. All German letters will denote integral ideals of K, especially $\mathfrak{p}$ prime ideals. All estimates in $\ll$ or O are independent to $\mathfrak{f}$ and χ. All Latin letters will denote some constants.

3. Preliminary from the theory of functions and numbers. To begin with, we quote the functional equation of $L(s,\chi)$, where χ is primitive.

Theorem A. *We assume that χ is primitive. Let put*

$$A(\mathfrak{f})=2^{-r_2}\pi^{-n/2}\sqrt{|d|\,N(\mathfrak{f})},$$

where d is a discriminant of K, and

$$\xi(s,\chi)=A(\mathfrak{f})^s\Gamma\left(\frac{s+1}{2}\right)^q\Gamma\left(\frac{s}{2}\right)^{r_1-q}\Gamma(s)^{r_2}L(s,\chi). \tag{1}$$

Then we have

$$\xi(s,\chi)=W(\chi)\xi(1-s,\bar{\chi}),$$

where $W(\chi)$ is only determined by χ and $|W(\chi)|=1$.

From this theorem we obtained the following estimate.

Proposition 1. *If $-10\leqq\sigma<0$, then we have*

$$|L(s,\chi)|\leqq c_1(\sigma)(N(\mathfrak{f})(|t|+2)^n)^{-(\sigma-1/2)}. \tag{2}$$

Proof. We first assume that χ is primitive. Then, by Theorem A, Stirling's formula and the assumption, we have

$$\begin{aligned}|L(s,\chi)|&\leqq cA(\mathfrak{f})^{1-2\sigma}(|t|+2)^{-n(\sigma-1/2)}\\&\leqq c_1(\sigma)(N(\mathfrak{f})(|t|+2)^n)^{-(\sigma-1/2)}.\end{aligned}$$

If χ is not primitive, then let χ^* be a primitive character induced by χ. We assume that χ^* is defined mod $\mathfrak{f}^*$. Then, by the definition of $L(s,\chi)$, we get

$$L(s,\chi)=L(s,\chi^*)\prod_{\substack{\mathfrak{p},\chi^*(\mathfrak{p})\neq 0\\ \chi(\mathfrak{p})=0}}\left(1-\frac{\chi^*(\mathfrak{p})}{N(\mathfrak{p})^s}\right)$$

and

$$\begin{aligned}\left|\prod_{\mathfrak{p},\chi^*(\mathfrak{p})\neq 0,\chi(\mathfrak{p})=0}\left(1-\frac{\chi^*(\mathfrak{p})}{N(\mathfrak{p})^s}\right)\right|&\leqq c(\sigma)\prod_{\mathfrak{p}|\mathfrak{f}/\mathfrak{f}^*}N(\mathfrak{p})^{-\sigma+1/2}\\&\leqq c_1(\sigma)N(\mathfrak{f}/\mathfrak{f}^*)^{-\sigma+1/2}.\end{aligned}$$

Hence

$$|L(s,\chi)|\leqq c_1(\sigma)(N(\mathfrak{f}^*)(|t|+2)^n)^{-\sigma+1/2}N(\mathfrak{f}/\mathfrak{f}^*)^{-\sigma+1/2}.$$

q.e.d.

Corollary 2. *For $s=\sigma+it$, $-\varepsilon\leqq\sigma\leqq 1+\varepsilon$, we get*

$$|(s-1)L(s,\chi)|\leqq c_2(\varepsilon)(N(\mathfrak{f})(|t|+2)^n)^{(1+\varepsilon-\sigma)/2}(|t+1|). \tag{3}$$

This corollary is immediately obtained by Proposition 1 and Phragmén-Lindelöf principle. Now we quote two well-known theorems;

Proposition 3. *Let Z denote the number of zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ which satisfy the conditions $\beta\geqq 0$ and $t\leqq\gamma\leqq t+1$. Then we have*

$$Z\leqq c_3\log N(\mathfrak{f})(|t|+2)^n. \tag{4}$$

Proposition 4. *Let $f(s)$ be regular in the disc $|s-s_0|\leqq r$, and $f(s_0)\neq 0$. We assume that $|f(s)/f(s_0)|<e^M$ there. And ρ runs over the zeros of $f(s)$ in the region $|s-s_0|\leqq r/2$. Then for s, $|s-s_0|\leqq r/4$, we have*

$$\left|\frac{f'}{f}(s)-\sum_\rho\frac{1}{s-\rho}\right|<\frac{16M}{r}. \tag{5}$$

Proposition 5. *For s, $\left|s-\left(\frac{9}{8}+it\right)\right|\leqq 1/4$ (and $|t|\geqq 2$ if $\chi=\chi_0$), we have*

$$\left|\frac{L'}{L}(s,\chi)-\sum_\rho\frac{1}{s-\rho}\right|\leqq 8\log N(\mathfrak{f})(|t|+2)^n+O(1) \tag{6}$$

where ρ runs over the zeros of $L(s,\chi)$ in the disc $\left|s-\frac{9}{8}+it\right|\leqq 1/2$.

This is easily obtained, putting $r=1$ and $s_0=\frac{9}{8}+it$ in Proposition 4 and $\varepsilon=1/8$ in Corollary 2.

Proposition 6 (Linnik). *Let $0<\delta<1/8$ and let N denote the number of zeros of $L(s,\chi)$ in the region $|s-(1+it)|<\delta$. Then we have*

$$N\leqq 32\delta\log N(\mathfrak{f})(|t|+2)^n+O(1). \tag{7}$$

This proposition is obtained by the same method as Lemma 2.1 in Prachar [3] Chap. X, applying Proposition 5.

Proposition 7. *We define the function $p_k(u)=e^{-u}\frac{u^k}{k!}$. Then we have*

$$0\leqq p_k(u)\leqq 1 \qquad \text{for } u\geqq 0,$$
$$p_k(u)\leqq b_1^k e^k \quad \text{for } 0\leqq u\leqq b_1 k, \quad \text{where } b_1<1,$$

and

$$p_k(u)\leqq (e^{1-b_2/2}b_2)^k e^{-u/2} \quad \text{for } u\geqq b_2 k, \quad \text{where } b_2>2.$$

The proof is immediately done. (See also Gallagher [loc. cit.] p. 335.) Finally we quote more two famous theorems;

Theorem B (Turán). *Let $z_1, z_2, \cdots, z_k$ be complex numbers satisfying*

$$1\geqq|z_1|\geqq|z_2|\geqq\cdots\geqq|z_k|.$$

Then for all positive integers m and n ($n\geqq k$), we have

$$\operatorname*{Max}_{m\leqq\nu\leqq m+n}\left|\sum_{j=1}^{k} z_j^\nu\right|\geqq\left(\frac{n}{8e(m+n)}\right)^n.$$

Theorem C (Brun-Titchmarsh). *Let $\mathfrak{C}$ be one of ideal classes* mod $\tilde{\mathfrak{f}}$. *For $x\geqq(N(\mathfrak{f})/\varepsilon^n)^2$, where $0<\varepsilon<1$, we have*

$$\pi(xe^\varepsilon,\mathfrak{C})-\pi(x,\mathfrak{C})\ll\frac{\varepsilon x}{h(\tilde{\mathfrak{f}})\log x},$$

uniformly on ε, x and $\mathfrak{f}$, where $\pi(x,\mathfrak{C})=\sum_{N(\mathfrak{p})\leqq x,\mathfrak{p}\in\mathfrak{C}} 1$ and $h(\tilde{\mathfrak{f}})$ denote the ideal class number mod $\tilde{\mathfrak{f}}$.

4. The proof of the density estimate. We put $w=1+iv$, $|v|\leqq T$ (and if $\chi=\chi_0$, we also assume that $|v|\geqq 2$), and $\mathcal{L}=\log N(\mathfrak{f})T^n$. We define the function

$$S_{x,y}(\chi,v)=\sum_{x\leqq N(\mathfrak{p})<y}\frac{\chi(\mathfrak{p})\log N(\mathfrak{p})}{N(\mathfrak{p})^w}.$$

Theorem 8. *If $L(s,\chi)$ has a zero in the disc $|s-w|\leqq r$ with $c_4\mathcal{L}^{-1}\leqq r\leqq r_0$, then for each $x\geqq e^{A\mathcal{L}}$, we have*

$$\int_x^{x^B}|S_{x,y}(\chi,v)|\frac{dy}{y}\gg x^{-cr}\log^2 x. \tag{*}$$

(The constants in this theorem will be defined in the proof.)

Proof. By Proposition 5 and Cauchy's integration formula, for s, $\left|s-\left(\frac{9}{8}+iv\right)\right|\leqq 1/8$, we have

$$\left|\frac{1}{k!}\frac{d^k}{ds^k}\frac{L'}{L}(s,\chi)-(-1)^k\sum_{|\rho-(9/8+iv)|\leqq 1/2}\frac{1}{(s-\rho)^{k+1}}\right|<c_5 8^k\mathcal{L}.$$

We now put $s=br+w$, where $r<1/8$ and $br<1/4$, and estimate the contribution of the terms with $|\rho-w|>\lambda$, where $r\leqq\lambda\leqq 1/8$. Since we know from Propositions 3 and 6 that there are at most $c_6 2^j\lambda\mathcal{L}$ terms satisfying the condition $2^j\lambda\leqq|\rho-w|<2^{j+1}\lambda$ and each of such terms is at most $(2^j\lambda)^{-(k+1)}$, then for $k\geqq 1$, the contribution is at most

$$\sum_{j=0}^{\infty} c_6 2^j\lambda\mathcal{L}(2^j\lambda)^{-(k+1)}<2c_6\lambda^{-k}\mathcal{L}.$$

Therefore we obtain

$$\left|\frac{1}{k!}\frac{d^k}{ds^k}\frac{L'}{L}(s,\chi)-(-1)^k\sum_{|\rho-w|\leqq\lambda}\frac{1}{(s-\rho)^{k+1}}\right|<c_7\lambda^{-k}\mathcal{L},$$

where $c_7=c_5+2c_6$. There are at most $(32+\varepsilon_1)\,\lambda\mathcal{L}$ zeros with $|\rho-w|\leqq\lambda$ and by the assumption $\operatorname*{Min}_{|\rho-w|\leqq\lambda}|s-\rho|\leqq(1+b)r$. Hence, from Theorem B, we have

$$\left|\sum_{|w-\rho|\leqq\lambda}\frac{1}{(s-\rho)^{k+1}}\right|\geqq\left(\frac{1}{8e(l+1)}\right)^K\left(\frac{1}{(1+b)r}\right)^{k+1}$$

for some integer $k\in[lK-1,(l+1)K-1]$ under the assumption

$$K\geqq(32+\varepsilon_1)\lambda\mathcal{L}. \tag{8}$$

But

$$\left(\frac{1}{8e(l+1)}\right)^K\geqq\left(\frac{1}{8e(l+1)}\right)^{(k+1)/l}=F^{-(k+1)},\text{ say.}$$

Hence we get

$$\left|\sum_{|w-\rho|\leqq\lambda}\frac{1}{(s-\rho)^{k+1}}\right|\geqq\left(\frac{1}{(1+b)Fr}\right)^{k+1}.$$

Putting $\lambda=ar$, we consider the inequality

$$\left|\left(\frac{1}{(1+b)Fr}\right)^{k+1}\right|\geqq 2c_7\lambda^{-k}\mathcal{L}.$$

The condition is satisfied if we choose $a=(1+b)(1+\varepsilon_2)F$ and assume $r\mathcal{L}\geqq c(\varepsilon_2)$ because

$$(1+\varepsilon_2)^k\geqq e^{\varepsilon_2/2\cdot lK}\geqq 4Fc_7 r\mathcal{L}.$$

Under the assumption, we get

$$\left|\frac{1}{k!}\frac{L'}{L}(s,\chi)\right|\geqq\frac{1}{2}\left(\frac{1}{(1+b)Fr}\right)^{k+1}.$$

Because of Re $s>1$ and $s=w+br$, we have

$$\left|\sum_{\mathfrak{a}}\frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a})}{N(\mathfrak{a})^w}p_k(br\log N(\mathfrak{a}))\right|\geqq\frac{1}{2b}\,\frac{((1+b^{-1})F)^{-(k+1)}}{r}. \tag{9}$$

We put $A=(1+b)B_1(32+\varepsilon_1)(1+\varepsilon_2)Fl$ and for given $x\geqq(N(\mathfrak{f})T^n)^A$, we also put $K=B_1^{-1}r\log x$ and $B=(l+1)B_2/B_1$, where B_2 is sufficiently large. From this selection, the assumption (8) is satisfied and for $N(\mathfrak{a})<x$, we have $p_k(br\log N(\mathfrak{a}))\leqq(bB_1e)^k$ and for $N(\mathfrak{a})\geqq x^B$, $p_k(br\log N(\mathfrak{a}))\leqq(beB_2/e^{bB_2/2})^kN(\mathfrak{a})^{-br/2}$. Hence we get

$$\left|\sum_{N(\mathfrak{a})<x}\frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a})}{N(\mathfrak{a})^w}p_k(br\log N(\mathfrak{a}))\right|\leqq 2(bB_1e)^k\frac{B_1k}{r}$$

and

$$\left|\sum_{N(\mathfrak{a})\geqq x^B}\frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a})}{N(\mathfrak{a})^w}p_k(br\log N(\mathfrak{a}))\right|\leqq 4(bB_2e^{1-bB_2/2})^k/r.$$

Taking $B_1=1/(1+b)Fe^2$, we have

$$|2(beB_1)^k|\leqq\frac{1}{16b}((1+b^{-1})Fr)^{-(k+1)}$$

and

$$|4(bB_2e^{1-bB_2/2})^k|<\frac{1}{16b}((1+b^{-1})F)^{-(k+1)}.$$

Collecting these results, we have

$$\left|\sum_{x\leqq N(\mathfrak{a})<x^B}\frac{\chi(\mathfrak{a})\Lambda(\mathfrak{a})}{N(\mathfrak{a})^w}p_k(br\log N(\mathfrak{a}))\right|\geqq\frac{1}{4b}((1+b^{-1}F)^{-(k+1)/r}. \tag{10}$$

The contribution to (10) of the terms of prime ideal powers $\mathfrak{a}=\mathfrak{p}^j$, $j\geqq 2$, is $O(x^{-1/2})$, which is smaller than half of the right hand side of (10), assuming that r_0 is sufficiently small. By the partial summation, we have

$$\sum_{x\leqq N(\mathfrak{p})<x^B}\frac{\chi(\mathfrak{p})\log N(\mathfrak{p})}{N(\mathfrak{p})^w}p_k(br\log N(\mathfrak{p}))$$

$$=p_k(br\log x^B)S_{x,x^B}(\chi,v)-\int_x^{x^B}S_{x,y}(\chi,v)p_k'(br\log y)\frac{br}{y}dy.$$

The first term of the right hand side is at most

$$2(bB_2e^{1-bB_2/2})^kB\log x\leqq\frac{1}{32b}\,\frac{((1+b^{-1})F)^{-(k+1)}}{r}.$$

Since $|p_k'(u)|\leqq 2$, we have

$$\int_x^{x^B}|S_{x,y}(\chi,v)|\frac{dy}{y}\gg\frac{((1+b^{-1})F)^{-(k+1)}}{r^2}$$

$$\gg\frac{\log^2 x}{(lK)^2}((1+b^{-1})F)^{-(l+1)K}$$

$$\gg x^{-((1+\varepsilon_3)+1/l)B_1^{-1}r\log(1+b^{-1})F}\log^2 x,$$

which is the assertion of the theorem. q.e.d.

Theorem 9. *We have*

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}}N(\sigma,T,\chi)\ll(N(\mathfrak{f})T^n)^{c_0(1-\sigma)},$$

where $c_0=2890$.

Proof. We may assume that $c_4\mathcal{L}^{-1}\leqq 1-\sigma\leqq\log\mathcal{L}/\mathcal{L}$, because we know

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} N(\sigma, T, \chi) \ll (N(\mathfrak{f})T^n)^{c_8(1-\sigma)} \log^{c_9} N(\mathfrak{f})T^n,$$

where $c_8 + c_9 \leqq c_0$ and the right hand side is essentially constant for $0 \leqq 1-\sigma < c_4 \mathcal{L}^{-1}$. We may also not consider the zeros of $L(s, \chi_0)$ in $0<\sigma<1$ and $|t|<2$. Put $r=(1+\varepsilon_4)(1-\sigma)$. By Theorem 8, if there exists a zero in the disc $|s-w| \leqq r$, then for $x \geqq (N(\mathfrak{f})T^n)^A$, *) is true. We now assume $l \geqq 1$. Then $A \geqq 2$ and we choose $x=(N(\mathfrak{f})T^n)^A$. Using the Schwarz inequality, we get ($c^* = 2(1+\varepsilon_4)Ac$)

$$1 \ll (N(\mathfrak{f})T^n)^{c^*(1-\sigma)} \mathcal{L}^{-3} \int_x^{x^B} |S_{x,y}(\chi, v)|^2 \frac{dy}{y}.$$

By the same argument in Gallagher (loc. cit.), we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} N(\sigma, T, \chi) \ll (N(\mathfrak{f})T^n)^{c^*(1-\sigma)} \mathcal{L}^{-2} \int_x^{x^B} \sum_{\chi \bmod \tilde{\mathfrak{f}}} |S_{x,y}(\chi, v)|^2 dv \frac{dy}{y}.$$

Using Gallagher's mean value theorem (Theorem 1 in Gallagher [loc. cit.]) we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \int_{-T}^{T} |S_{x,y}(\chi, v)|^2 dv \ll T^2 \int_x^{x^B} \sum_{\chi \bmod \tilde{\mathfrak{f}}} \left| \sum_{y \leqq N(\mathfrak{p}) < y\tau} \chi(\mathfrak{p}) a_{\mathfrak{p}} \right|^2 \frac{dy}{y}$$

where $a_{\mathfrak{p}} = \log N(\mathfrak{p})/N(\mathfrak{p})$ if $x \leqq N(\mathfrak{p}) < x^B$ and 0 if x is otherwise and $\tau = e^{1/T}$. By the orthogonal relation Cauchy-Schwarz inequality and Theorem C, we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \left| \sum_{y \leqq N(\mathfrak{p}) < y\tau} \chi(\mathfrak{p}) a_{\mathfrak{p}} \right|^2 = h(\tilde{\mathfrak{f}}) \sum_{\mathfrak{C} \bmod \tilde{\mathfrak{f}}} \left| \sum_{\substack{y \leqq N(\mathfrak{p}) < y\tau \\ \mathfrak{p} \in \mathfrak{C}}} a_{\mathfrak{p}} \right|^2$$
$$\ll \frac{1}{T} \frac{y}{\log x} \sum_{y \leqq N(\mathfrak{p}) < y\tau} \frac{\log^2 N(p)}{N(p)^2}$$

Hence, we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} N(\sigma, T, \chi) \ll (N(\mathfrak{f})T^n)^{c^*(1-\sigma)},$$

with $c^* = 2(1+\varepsilon_4)Ac = 2890$, taking $b=(2 \log F)^{-1/2}$ and $l=9$. q.e.d.

References

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