3. *The Fundamental Solution for a Degenerate Parabolic Pseudo-Differential Operator*

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(Comm. by Kôsaku YOSIDA, M. J. A., Jan. 12, 1974)

Introduction. In the present paper we shall construct the fundamental solution $U(t)$ for a degenerate parabolic pseudo-differential equation of the form

$$\begin{cases} Lu=\dfrac{\partial u}{\partial t}+p(t\,;x,D)u=0 & \text{in } (0,T)\times R^n \\ u|_{t=0}=u_0 \end{cases} \tag{0.1}$$

where $p(t\,;x,D)$ is a pseudo-differential operator of class $\mathcal{E}_t^0(S^m_{\rho,\delta})$ which satisfies conditions (cf. [1], [5]):

(i) There exist constant C and m' $(0\leqq m'\leqq m)$ such that

$$\operatorname{Re} p(t\,;x,\xi)\geqq C\langle\xi\rangle^{m'} \qquad \text{uniformly in } t \quad (0\leqq t\leqq T). \tag{0.2}$$

(ii) For any multi index $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_n)$ there exists a constant $C_{\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(t\,;x,\xi)/\operatorname{Re} p(t\,;x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \qquad \text{uniformly in } t \quad (0\leqq t\leqq T), \tag{0.3}$$

where $p^{(\alpha)}_{(\beta)}(t\,;x,\xi)=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_n)^{\alpha_n}(-i\partial/\partial x_1)^{\beta_1}\cdots(-i\partial/\partial x_n)^{\beta_n}p(t\,;x,\xi)$, $|\alpha|=|\alpha_1|+\cdots+|\alpha_n|$, $|\beta|=|\beta_1|+\cdots+|\beta_n|$ and $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$.

The fundamental solution $U(t)$ will be found as a pseudo-differential operator of class $S^0_{\rho,\delta}$ with parameter t. Then the solution of the Cauchy problem (0.1) is given by $u(t)=U(t)u_0$ for $u_0\in L^2$ and moreover for $u_0\in L^p$ $(1<p<\infty)$ in case $\rho=1$, using that operators of class $S^m_{\rho,\delta}$ are bounded in L^2 for $0\leqq\delta<\rho\leqq1$, in L^p for $0\leqq\delta<1$, $\rho=1$ (see [1]–[3]).

The solution $U(t)$ is given in the form $U(t)=e(t,0\,;x,D)$ where $e(t,s\,;x,D)$ is the solution of an operator equation

$$\begin{cases} L_{x,t}e(t,s\,;x,D)=0 & \text{in } t>s \quad (0\leqq s<t\leqq T) \\ e(t,s\,;x,D)|_{t=s}=I, \end{cases}$$

which can be reduced to an integral equation of the form

$$r_N(t,s\,;x,D)+\varphi(t,s\,;x,D)+\int_s^t r_N(t,\sigma\,;x,D)\varphi(\sigma,s\,;x,D)d\sigma=0, \tag{0.4}$$

where $r_N(t,s\,;x,D)$ is a known operator of class $S^{m-(\rho-\delta)(N+1)}_{\rho,\delta}$. To solve (0.4), we shall calculate the symbol for multi product of pseudo-differential operators in precise form by using oscillatory integrals in [4] and [6].

1. Notations and Theorem. We shall denote by $S^m_{\rho,\delta}$ $(0\leqq\delta<\rho\leqq1,$

$-\infty < m < \infty$) the set of all C^∞-symbols $p(x, \xi)$ defined in $R^n_x \times R^n_\xi$, which satisfy for multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C'_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{1.1}$$

for some constants $C'_{\alpha,\beta}$, where $p^{(\alpha)}_{(\beta)}(x, \xi)$ is defined as above. For a symbol $p(x, \xi) \in S^m_{\rho,\delta}$ we define a pseudo-differential operator by

$$Pu(x) = p(x, D)u(x) = \int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi,$$

where $d\xi = (2\pi)^{-n}d\xi$ and $\hat{u}(\xi)$ denotes the Fourier transform of a rapidly decreasing function $u(x)$ defined by

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx.$$

Definition 1.1. For a $p(x, \xi) \in S^m_{\rho,\delta}$ we define semi-norms $|p|_{m,k}$ by

$$|p|_{m,k} = \max_{|\alpha|+|\beta|\leqq k} \sup_{(x,\xi)} \{|p^{(\alpha)}_{(\beta)}(x, \xi)|\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}\}$$

then, $S^m_{\rho,\delta}$ makes a Fréchet space with these norms. $\mathcal{E}^0_t(S^m_{\rho,\delta})$ is the set of all functions $p(t; x, \xi)$ of class $S^m_{\rho,\delta}$ which are continuous with respect to parameter t for $0 \leqq t \leqq T$.

Definition 1.2 ([5]). We say $\{p_j(x, \xi)\}^\infty_{j=0}$ of $S^m_{\rho,\delta}$ converges to a $p(x, \xi) \in S^m_{\rho,\delta}$, weakly, if $\{p(x, \xi)\}^\infty_{j=0}$ is a bounded set of $S^m_{\rho,\delta}$ and $p_j{}^{(\alpha)}_{(\beta)}(x, \xi) \to p^{(\alpha)}_{(\beta)}(x, \xi)$ as $j \to \infty$ uniformly on $R^n_x \times K$ for any α, β, where K is any compact set in R^n. We denote by $w-\mathcal{E}^0_{t,s}(S^m_{\rho,\delta})$ the set of all functions $p(t, s; x, \xi)$ of class $S^m_{\rho,\delta}$ $(0 \leqq s \leqq t \leqq T)$ which are continuous with respect to parameters t and s with weak topology of $S^m_{\rho,\delta}$.

Theorem. *Under the assumptions* (0.2) *and* (0.3) *we can construct* $E(t, s) = e(t, s; x, D) \in w-\mathcal{E}^0_{t,s}(S^m_{\rho,\delta})$ $(0 \leqq s \leqq t \leqq T)$ *which satisfies the following conditions*:

(A) $L_{x,t}E(t, s) = 0$ *in* $t > s$

(B) $E(t, s)|_{t=s} = I$

(C) *For any sufficiently large* N, *we can write*

$$e(t, s; x, \xi) = \sum_{j=0}^{N} e_j(t, s; x, \xi) + (t-s)f_N(t, s; x, \xi)$$

where

(C–1) $e_j(t, s; x, \xi) \in w-\mathcal{E}^0_{t,s}(S^{-(\rho-\delta)j}_{\rho,\delta})$

(C–2) $e_0(t, s; x, \xi) \to 1$ $(t \downarrow s)$ *in* $S^0_{\rho,\delta}$ *weakly*

(C–3) $e_j(t, s; x, \xi) \to 0$ $(t \downarrow s)$ *in* $S^{-(\rho-\delta)j}_{\rho,\delta}$ *weakly* $(j \geqq 1)$

(C–4) $f_N(t, s; x, \xi) \in w-\mathcal{E}^0_{t,s}(S^{m-(\rho-\delta)(N+1)}_{\rho,\delta})$

(C–5) $|f_N{}^{(\alpha)}_{(\beta)}(t, s; x, \xi)| \leqq C_{\alpha,\beta}(t-s)\langle\xi\rangle^{2m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}$ *for any* α, β.

2. Proof of Theorem.

As in [8], [7], we construct $e_j(t, s; x, \xi)$ $(0 \leqq s \leqq t \leqq T)$ $(j \geqq 0)$ in the following way.

$$\begin{cases} \left[\dfrac{\partial}{\partial t} + p(t; x, \xi)\right]e_0(t, s; x, \xi) = 0 & \text{in } t > s \\ e_0(t, s; x, \xi)|_{t=s} = 1 \end{cases} \tag{2.1}$$

and for $j \geqq 1$

$$(2.2)\quad \begin{cases} \left[\dfrac{\partial}{\partial t}+p(t;x,\xi)\right]e_j(t,s;x,\xi)=-q_j(t,s;x,\xi) & \text{in } t>s \\ e_j(t,s;x,\xi)|_{t=s}=0, \end{cases}$$

where $q_j(t,s;x,\xi)$ is defined by

$$(2.3)\qquad q_j(t,s;x,\xi)=\sum_{k=0}^{j-1}\sum_{|\alpha|+k=j}\frac{1}{\alpha!}p^{(\alpha)}(t;x,\xi)e_{k(\alpha)}(t,s;x,\xi).$$

Set $e_{j}{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)=a_{j,\alpha,\beta}(t,s;x,\xi)\exp\left(-\int_s^t p(\sigma;x,\xi)d\sigma\right)$ ($j\geqq 0$) and $q_{j}{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)=b_{j,\alpha,\beta}(t,s;x,\xi)\exp\left(-\int_s^t p(\sigma;x,\xi)d\sigma\right)$ ($j\geqq 1$). Then we have by (2.1)$\sim$(2.3) and (0.3) the following estimates.

Proposition 1. *We have*

$$|a_{j,\alpha,\beta}(t,s;x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j}\omega_{j,\alpha,\beta},$$
$$|b_{j,\alpha,\beta}(t,s;x,\xi)|\leqq C_{\alpha,\beta}\,\mathrm{Re}\,p(t;x,\xi)\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j}\omega'_{j,\alpha,\beta}$$

where $\omega_{j,\alpha,\beta}$ and $\omega'_{j,\alpha,\beta}$ are defined by

$$\omega_{0,0,0}=1,\qquad \omega_{0,\alpha,\beta}=\max\{\omega,\omega^{|\alpha|+|\beta|}\}\ |\alpha|+|\beta|\neq 0$$
$$\omega_{j,\alpha,\beta}=\max\{\omega^2,\omega^{|\alpha|+|\beta|+2j}\}\qquad (j\geqq 1)$$
$$\omega'_{j,\alpha,\beta}=\max\{\omega,\omega^{|\alpha|+|\beta|+2j-1}\}\qquad (j\geqq 1)$$

and $\omega=\int_s^t \mathrm{Re}\,p(\sigma;x,\xi)d\sigma$.

Now by the expansion theorem in [2], we can write for any N

$$(2.4)\quad \sigma(PE_j)=p(t;x,\xi)e_j(t,s;x,\xi)+\sum_{0<|\alpha|\leqq N-j}\frac{1}{\alpha!}p^{(\alpha)}(t;x,\xi)\times e_{j(\alpha)}(t,s;x,\xi)+r_{N,j}(t,s;x,\xi).$$

Taking summation in j, it is clear by (2.1)$\sim$(2.3) that

$$(2.5)\quad \begin{aligned} L_{x,t}\left(\sum_{j=0}^N E_j\right)&=\sum_{j=0}^N\left[\left(\frac{\partial}{\partial t}+p\right)e_j\right](t,s;x,D)+\sum_{j=1}^N q_j(t,s;x,D)\\ &\quad+\sum_{j=0}^N r_{N,j}(t,s;x,D)\\ &=\sum_{j=0}^N r_{N,j}(t,s;x,D)\equiv r_N(t,s;x,D). \end{aligned}$$

The following estimates are clear with the aid of Proposition 1 and (2.4).

Proposition 2. *We have $r_{N,j}(t,s;x,\xi)\in w-\mathcal{E}^0_{t,s}(S^{m-(\rho-\delta)(N+1)}_{\rho,\delta})$ and for any α,β*

$$|r_{N,j}{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqq C_{\alpha,\beta}(t-s)\langle\xi\rangle^{2m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}.$$

Put $\sum_{j=0}^N e_j(t,s;x,D)=k_N(t,s;x,D)$, then we have by (2.5)

$$(2.6)\quad \begin{cases} L_{x,t}K_N(t,s)=R_N(t,s) & \text{in } t>s\quad (0\leqq s<t\leqq T)\\ K_N(t,s)|_{t=s}=I. \end{cases}$$

Now, we construct $e(t,s;x,D)$ as the following form:

$$e(t,s;x,D)=k_N(t,s;x,D)+\int_s^t k_N(t,\sigma;x,D)\varphi(\sigma,s;x,D)d\sigma.$$

Then, using (2.6), $\varphi(t,s;x,D)=\Phi(t,s)$ must satisfy

$$L_{x,t}E(t,s)=R_N(t,s)+\Phi(t,s)+\int_s^t R_N(t,\sigma)\Phi(\sigma,s)d\sigma. \tag{2.7}$$

Set

$$\Phi_1(t,s)=-R_N(t,s),$$

and for $j\geqq 2$

$$\begin{aligned}\Phi_j(t,s)&=\int_s^t \Phi_1(t,\sigma)\Phi_{j-1}(\sigma,s)d\sigma\\&=\int_s^t\int_s^{s_1}\cdots\int_s^{s_{j-2}}\Phi_1(t,s_1)\Phi_1(s_1,s_2)\Phi_1(s_2,s_3)\\&\qquad\cdots\Phi_1(s_{j-1},s)ds_{j-1}ds_{j-2}\cdots ds_1.\end{aligned} \tag{2.8}$$

Then

$$\begin{aligned}\sum_{j=1}^{l}\Phi_j(t,s)&=\Phi_1(t,s)+\sum_{j=2}^{l}\Phi_j(t,s)\\&=-R_N(t,s)-\int_s^t R_N(t,\sigma)\sum_{j=1}^{l-1}\Phi_j(\sigma,s)d\sigma.\end{aligned} \tag{2.9}$$

For $\sigma(\Phi_j(t,s))=\varphi_j(t,s;x,\xi)$, we have the following

Proposition 3. *We have some constants $A_{\alpha,\beta}$ and $A'_{\alpha,\beta}$, which are independent of j, such that*

$$|\varphi_{j}{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqq(A_{\alpha,\beta})^j\frac{(t-s)^{j-1}}{(j-1)!}(t-s)\langle\xi\rangle^{2m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(N+1)} \tag{2.10}$$

$$|\varphi_{j}{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqq(A'_{\alpha,\beta})^j\frac{(t-s)^{j-1}}{(j-1)!}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(N+1)}. \tag{2.11}$$

In view of Proposition 3, we have $\sum_{j=1}^{\infty}\varphi_j=\varphi\in w-\mathcal{E}^0_{t,s}(S^{m-(\rho-\delta)(N+1)}_{\rho,\delta})$ and (2.9) means that $\Phi(t,s)=\varphi(t,s;x,D)$ given above satisfies (2.7). Note that $K_N(t,s)\in w-\mathcal{E}^0_{t,s}(S^0_{\rho,\delta})$ and

$$|\varphi^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqq C_{\alpha,\beta}(t-s)\langle\xi\rangle^{2m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(N+1)}.$$

Then we have the assertion of theorem.

Proof of Proposition 3. Using the oscillatory integral in [4], we have from (2.8)

$$\begin{aligned}\varphi_j(t,s;x,\xi)=\int_s^t\int_s^{s_1}\cdots\int_s^{s_{j-2}}ds_{j-1}\cdots ds_1\Big[O_s-\iint\cdots\int e^{-i\sum_{l=1}^{j-1}\eta_l\cdot y_l}\\\times\varphi_1(t,s_1;x,\xi+\eta_1)\prod_{k=1}^{j-2}\varphi_1\Big(s_k,s_{k+1};x+\sum_{l=1}^{k}y_l,\xi+\eta_{k+1}\Big)\\\times\varphi_1\Big(s_{j-1},s;x+\sum_{l=1}^{j-1}y_l,\xi\Big)dy_1d\eta_1\cdots dy_{j-1}d\eta_{j-1}\Big].\end{aligned}$$

Note $\varphi_1\in w-\mathcal{E}^0_{t,s}(S^{m-(\rho-\delta)(N+1)}_{\rho,\delta})$ and rewrite

$$e^{-iy_k\cdot\eta_k}=(1+\langle\xi+\eta_k\rangle^{2n_0\delta}|y_k|^{2n_0})^{-1}(1+\langle\xi+\eta_k\rangle^{2n_0\delta}(-\Delta_{\eta_k})^{n_0})e^{-iy_k\cdot\eta_k}.$$

Then we have

$$\begin{aligned}|\varphi_j|\leqq(C_{n_0})^j\int_s^t\int_s^{s_1}\cdots\int_s^{s_{j-2}}ds_{j-1}\cdots ds_1\langle\xi\rangle^{m-(\rho-\delta)(N+1)}\\\times\prod_{k=1}^{j-1}\iint(1+\langle\xi+\eta_k\rangle^{2n_0\delta}|y_k|^{2n_0})^{-1}\langle\xi+\eta_k\rangle^{m-(\rho-\delta)(N+1)}dy_kd\eta_k,\end{aligned}$$

where $n_0>(n/2)$ is an integer. If we take N such that $m-(\rho-\delta)$

$(N+1)<-n$, then we get

$$|\varphi_j(t, s; x, \xi)| \leqq (C_{n_0})^j \frac{(t-s)^{j-1}}{(j-1)!} \langle \xi \rangle^{m-(\rho-\delta)(N+1)}.$$

By Proposition 2, we can prove (2.11) for $\alpha=\beta=0$. For any α, β (2.10) and (2.11) are proved in the same way.

Example.

$$L_{x,t} = \frac{\partial}{\partial t} + a(t)\,|x|^{2b}(-\Delta)^m + (-\Delta)$$

where $a(t) \in C^\infty[0, T]$, $a(t) \geqq 0$, and b and m are positive integers such that $b+1>m$.

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