1. *On the Degenerate Oblique Derivative Problems*

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§ 1. **Introduction and results.** In this note we give *a priori* estimates and existence theorems for the degenerate oblique derivative problems, which will be formulated below. We first reduce the given boundary value problems to the pseudo-differential equations on the boundary with the aid of suitable boundary value problems which are well studied, and next apply Melin's theorem (see [5], Theorem 3.1) to the pseudo-differential equations on the boundary.

Let Ω be a bounded domain in R^n, and we assume that $\Omega \cup \partial\Omega$ is a C^∞-manifold with boundary. Let $a(x)$, $b(x)$ and $c(x)$ be real-valued functions $\in C^\infty(\partial\Omega)$, $\boldsymbol{n}$ be the unit exterior normal to $\partial\Omega$ and $\boldsymbol{\nu}$ be a real C^∞-vector field on $\partial\Omega$.

Now we consider, for $\lambda>0$, the degenerate oblique derivative problem:

$$\text{(I)} \qquad \begin{cases} (\lambda-\Delta)u=f & \text{in } \Omega, \\ a(x)\dfrac{\partial u}{\partial \boldsymbol{n}}+b(x)\dfrac{\partial u}{\partial \boldsymbol{\nu}}+c(x)u=0 & \text{on } \partial\Omega, \end{cases}$$

under the following assumptions:

(1) $a(x)\geqq 0$.

(2) The set $S=\{x\in\partial\Omega\,;\,a(x)=0\}$ is an $(n-2)$-dimensional C^∞-manifold.

(3) $\boldsymbol{\nu}$ is transversal to S in $\partial\Omega$.

(4) $c(x)>0$ on the set $\{x\in\partial\Omega : a(x)=0\}$.

(5) Along the integral curve $x(t,x_0)$ of $\boldsymbol{\nu}$ passing $x_0\in S$ when $t=0$, $a(x(t,x_0))$ has a zero of finite order k at $t=0$, and $b(x(t,x_0))$ has a zero of finite order l at $t=0$, where k and l are independent of x_0.

Remark 1. In the case where $b(x)\neq 0$ on S, our problem is the oblique derivative problem which has been already treated by several authors and we can remove the assumption (4) (see [2] and [6]). In the case where $b(x)\equiv 0$, our problem was treated by S. Itô (see [3]) and we can also remove the assumptions (2) and (5) (see the proof below).

For each real s, we denote by $H_s(\Omega)$ and $H_s(\partial\Omega)$ the usual Sobolev spaces on Ω and $\partial\Omega$ respectively, and by $\|\ \|_{s,\Omega}$ and $\|\ \|_{s,\partial\Omega}$ norms in these spaces.

Theorem 1. *Assume that* $l\geqq k$ *and the assumptions* (1), (2), (4) *and* (5) *hold. Then there is a positive constant* C *such that, for* $u\in L^2(\Omega)$

satisfying (I) *with* $f \in L^2(\Omega)$ *we have*

(1.1) $$\|u\|_{2,\Omega} \leqq C(\|f\|_{0,\Omega} + \|u\|_{0,\Omega}).$$

Furthermore, if λ *is sufficiently large, there is a unique solution* $u \in H_2(\Omega)$ *for every* $f \in L^2(\Omega)$ *and we can omit* $\|u\|_{0,\Omega}$ *in the right hand side of* (1.1).

Theorem 2. *Assume that* $l<k$, l *is even and that the assumptions* (1), (2), (3), (4) *and* (5) *hold. Then there is a positive constant* C *such that, for* $u \in L^2(\Omega)$ *satisfying* (I) *with* $f \in L^2(\Omega)$ *we have*

(1.2) $$\|u\|_{1+\delta,\Omega} \leqq C(\|f\|_{0,\Omega} + \|u\|_{0,\Omega}), \qquad \text{where } \delta = \frac{1}{k-l+1}.$$

Furthermore, if λ *is sufficiently large, there is a unique solution* $u \in H_{1+\delta}(\Omega)$ *for every* $f \in L^2(\Omega)$, *and we can omit* $\|u\|_{0,\Omega}$ *in the right hand side of* (1.2).

§ 2. Sketch of proofs.

Proof of Theorem 1. First we consider the following problem:

(II) $$\begin{cases} (\lambda-\Delta)v=f & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} + \beta(x)\dfrac{\partial v}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\beta(x) \equiv \dfrac{b(x)}{a(x)}$ is naturally extended over $\partial\Omega$ by the assumption $l \geqq k$. For sufficiently large λ, there is a unique solution $v \in H_2(\Omega)$ satisfying (II) and

(2.1) $$\|v\|_{2,\Omega} \leqq C\,\|f\|_{0,\Omega}$$

where C is some positive constant.

The function $w=u-v$ satisfies

(III) $$\begin{cases} (\lambda-\Delta)w=0 & \text{in } \Omega, \\ a(x)\dfrac{\partial w}{\partial \boldsymbol{n}} + b(x)\dfrac{\partial w}{\partial \boldsymbol{\nu}} + c(x)w\,|_{\partial\Omega} = -c(x)v|_{\partial\Omega} \in H_{3/2}(\partial\Omega). \end{cases}$$

Recalling the fact that if we define π_λ by $\pi_\lambda\varphi = \left.\dfrac{\partial g}{\partial \boldsymbol{n}}\right|_{\partial\Omega}$ for $g \in L^2(\Omega)$ satisfying

(IV) $$\begin{cases} (\lambda-\Delta)g=0 & \text{in } \Omega, \\ \quad g|_{\partial\Omega} = \varphi \in H_{-1/2}(\partial\Omega), \end{cases}$$

then π_λ is a first order elliptic pseudo-differential operator on $\partial\Omega$, we can reduce the problem (III) to the following equation.

(V) $$T_\lambda w^0 \equiv a(x)\pi_\lambda w^0 + b(x)\frac{\partial w^0}{\partial \boldsymbol{\nu}} + c(x)w^0 = -c(x)v^0$$

where $w^0 = w|_{\partial\Omega}$. We now apply Melin's theorem to the operator $\Lambda^3 T_\lambda$, where $\Lambda = (1-\Delta')^{1/2}$ and Δ' is the Laplace-Beltrami operator on $\partial\Omega$. Since $a(x)$ vanishes at least with second order by (1), we can prove that Melin's conditions are satisfied for the operator $\Lambda^3 T_\lambda$. Hence we have

$$\operatorname{Re}(\Lambda^3 T_\lambda\phi, \phi) \geqq c_1\|\phi\|^2_{3/2,\partial\Omega} - c_2\|\phi\|^2_{t,\partial\Omega} \qquad \text{for any } \phi \in C^\infty(\partial\Omega)$$

where c_1 and c_2 are positive constants and $t<3/2$. On the other hand we have

$$\mathrm{Re}\,(\Lambda^3 T_\lambda\phi, \phi) \leqq c_3 \| T_\lambda\phi\|_{3/2,\partial\Omega} \|\phi\|_{3/2,\partial\Omega} \qquad \text{for any } \phi\in C^\infty(\partial\Omega).$$

Therefore

$$\|\phi\|_{3/2,\partial\Omega} \leqq c(\|T_\lambda\phi\|_{3/2,\partial\Omega} + \|\phi\|_{t,\partial\Omega})$$

for a suitable constant c. Hence we obtain $w^0\in H_{3/2}(\partial\Omega)$ and

$$\|w^0\|_{3/2,\partial\Omega} \leqq c(\|T_\lambda w^0\|_{3/2,\partial\Omega} + \|w^0\|_{t,\partial\Omega}). \tag{2.2}$$

Since w satisfies (IV) with φ replaced by $w^0\in H_{3/2}(\partial\Omega)$, we have

$$\|w\|_{2,\Omega} \leqq c\|w^0\|_{3/2,\partial\Omega}. \tag{2.3}$$

Hence, from (2.1), (2.2), (2.3) and $u=v+w$, we obtain

$$\|u\|_{2,\Omega} \leqq c(\|f\|_{0,\Omega} + \|u\|_{0,\Omega}).$$

We define the map $\mathcal{T}_\lambda : \mathcal{D}(\mathcal{T}_\lambda) \equiv \{\varphi\in H_{3/2}(\partial\Omega)\,;\, T_\lambda\varphi \in H_{3/2}(\partial\Omega)\} \to H_{3/2}(\partial\Omega)$ by $\mathcal{T}_\lambda\varphi = T_\lambda\varphi$ for $\varphi\in\mathcal{D}(\mathcal{T}_\lambda)$. Then the existence and the uniqueness of solution of the problem (I) are derived from those of the problem $\mathcal{T}_\lambda\varphi = \psi$. We apply Agmon's technique, that is, increasing the number of independent variables (see [1]), we obtain for large λ

$$\|u\|_{0,\Omega} \leqq \frac{c}{\lambda}\|(\lambda-\Delta)u\|_{0,\Omega}$$

for any u satisfying the boundary condition of (I). Hence the map $\mathcal{T}_\lambda$ must be one-to-one.

Next we consider

$$\begin{cases} (\lambda-\Delta)u = f & \text{in } \Omega, \\ a(x)\dfrac{\partial u}{\partial \boldsymbol{n}} + \left(b(x)\dfrac{\partial}{\partial \boldsymbol{\nu}}\right)^* u + c(x)u + ((a(x)\pi_\mu)^* - a(x)\pi_\mu)u\,|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

where $\mu\geqq 1$, and $P^*(x,D)$ denotes the formal adjoint of the pseudo-differential operator $P(x,D)$.

Considering the problem:

$$\begin{cases} (\lambda-\Delta)v = f & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} - \beta(x)\dfrac{\partial v}{\partial \boldsymbol{\nu}}\Big|_{\partial\Omega} = 0, \end{cases}$$

and setting $u=v+w$, we obtain the following pseudo-differential equation (2.5) on the boundary corresponding to (V):

$$a(x)\pi_\lambda w^0 + \left(b(x)\frac{\partial}{\partial \boldsymbol{\nu}}\right)^* w^0 + c(x)w^0 + ((a(x)\pi_\mu)^* - a(x)\pi_\mu)w^0 = g \tag{2.5}$$

where

$$g = -\left\{\left(b(x)\frac{\partial}{\partial \boldsymbol{\nu}} + \left(b(x)\frac{\partial}{\partial \boldsymbol{\nu}}\right)^*\right)v^0 + c(x)v^0 + ((a(x)\pi_\mu)^* - a(x)\pi_\mu)v^0\right\}.$$

We note that the left hand side of (2.5) equals T_λ^* when $\mu=\lambda$. It follows from the calculus for the pseudo-differential operators with parameter that there is a positive constant c_s independent of μ such that

$$\|((a(x)\pi_\mu)^* - a(x)\pi_\mu)\varphi\|_{s,\partial\Omega} \leqq c_s\|\varphi\|_{s,\partial\Omega} \qquad \text{for any } \varphi\in H_s(\partial\Omega).$$

Hence, by the same argument as above, we find a positive constant c independent of μ such that

$$\|u\|_{0,\Omega} \leqq \frac{c}{\lambda} \|(\lambda-\Delta)u\|_{0,\Omega}$$

for any u satisfying (2.4) and for large λ. We define the map $\mathcal{T}^*_{1,\lambda}$: $\mathcal{D}(\mathcal{T}^*_{1,\lambda}) \equiv \{\varphi \in H_{-3/2}(\partial\Omega)\,;\, T^*_\lambda\varphi \in H_{-3/2}(\partial\Omega)\} \to H_{-3/2}(\partial\Omega)$ by $\mathcal{T}^*_{1,\lambda}\varphi = T^*_\lambda\varphi$ for $\varphi \in \mathcal{D}(\mathcal{T}^*_{1,\lambda})$. Then $\mathcal{T}^*_{1,\lambda}$ is one-to-one. If we denote by $\tilde{\mathcal{T}}^*_\lambda$ the adjoint of $\mathcal{T}_\lambda$ with respect to the pairing of $H_{3/2}(\partial\Omega)$ and $H_{-3/2}(\partial\Omega)$, then $\mathcal{T}^*_{1,\lambda} \supset \tilde{\mathcal{T}}^*_\lambda$. Hence $\mathcal{T}_\lambda$ is onto for large λ since $\mathcal{T}_\lambda$ has a closed range which consists with the orthogonal complement of the null space of $\tilde{\mathcal{T}}^*_\lambda$. Theorem 1 is thus proved.

Proof of Theorem 2. Considering (II)′ below in place of (II), one can prove Theorem 2 by the same argument as the proof of Theorem 1:

$$\text{(II)}' \qquad \begin{cases} (\lambda-\Delta)v = f & \text{in } \Omega, \\ \alpha(x)\dfrac{\partial v}{\partial \boldsymbol{n}} + \dfrac{\partial v}{\partial \boldsymbol{\nu}}\Big|_{\partial\Omega} = 0, \end{cases}$$

where $\alpha(x) \equiv \dfrac{a(x)}{b(x)}$ is naturally extended over $\partial\Omega$ by the assumption $l<k$. This oblique derivative problem has a unique solution $v \in H_{1+\delta}(\Omega)$ for large λ (see [4]).

Remark 2. For $s \geqq 1$, by considering $\Lambda^{2s+3}T_\lambda$ in place of $\Lambda^3 T_\lambda$, one can easily obtain *a priori* estimates

$$\|u\|_{s+2,\Omega} \leqq c(\|f\|_{s,\Omega} + \|u\|_{0,\Omega}) \qquad \text{in place of (1.1)}$$

and

$$\|u\|_{s+1+\delta,\Omega} \leqq c(\|f\|_{s,\Omega} + \|u\|_{0,\Omega}) \qquad \text{in place of (1.2).}$$

If we define $\mathcal{A}$ on $\mathcal{D}(\mathcal{A}) \equiv \left\{u \in L^2(\Omega)\,;\, (\lambda_0-\Delta)u \in L^2(\Omega),\, a(x)\dfrac{\partial u}{\partial \boldsymbol{n}} + b(x)\dfrac{\partial u}{\partial \boldsymbol{\nu}} + c(x)u|_{\partial\Omega} = 0\right\}$ by $\mathcal{A}u = -(\lambda_0-\Delta)u$ for $u \in \mathcal{D}(\mathcal{A})$, then by the same argument as the proofs of Theorems 1 and 2 one can prove that

Theorem 3. *The spectrum of $\mathcal{A}$ is discrete and the eigenvalues of $\mathcal{A}$ have finite multiplicity. Moreover $\lambda = re^{i\theta}$ $(0 \leqq \theta < 2\pi, \theta \neq \pi)$ is contained in the resolvent set of $\mathcal{A}$ if r is sufficiently large.*

Remark 3. In all the Theorems stated above one can replace Δ by $\sum_{i,j=1}^n a_{ij}(x)\dfrac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x)\dfrac{\partial}{\partial x_j} + d(x)$ where $a_{ij}(x), b_j(x)$ and $d(x)$ are functions $\in C^\infty(\bar{\Omega})$ and $a_{ij}(x)$'s are real-valued functions satisfying $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqq c\,|\xi|^2$ with some constant $C>0$.

References

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