46. A Remark on Almost-Continuous Mappings

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1. Introduction. In 1968, M. K. Singal and A. R. Singal [2] defined almost-continuous mappings as a generalization of continuous mappings. They obtained an extensive list of theorems about such a mapping, among them, the following two results were established:

Theorem A. *Let $f_\alpha: X_\alpha \to X_\alpha^*$ be almost-continuous for each $\alpha \in I$ and let $f: \Pi X_\alpha \to \Pi X_\alpha^*$ be defined by setting $f((x_\alpha))=(f_\alpha(x_\alpha))$ for each point $(x_\alpha) \in \Pi X_\alpha$. Then f is almost-continuous.*

Theorem B. *Let $h: X \to \Pi X_\alpha$ be almost-continuous. For each $\alpha \in I$, define $f_\alpha: X \to X_\alpha$ by setting $f_\alpha(x)=(h(x))_\alpha$. Then f_α is almost-continuous for all $\alpha \in I$.*

The purpose of the present note is to show that the converses of the above two theorems are also true. As the present author has a question in the proof of Theorem B, we shall give the another proof.

2. Definitions and notations. Let A be a subset of a topological space X. By $\mathrm{Cl}\,A$ and $\mathrm{Int}\,A$ we shall denote the closure of A and the interior of A in X respectively. Moreover, A is said to be regularly open if $A=\mathrm{Int}\,\mathrm{Cl}\,A$, and regularly closed if $A=\mathrm{Cl}\,\mathrm{Int}\,A$. By a space we shall mean a topological space on which any separation axiom is not assumed. A mapping f of a space X into a space Y is said to be *almost-continuous* (simply *a.c.*) if for each point $x \in X$ and any neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x such that $f(U) \subset \mathrm{Int}\,\mathrm{Cl}\,V$. It is a characterization of *a.c.* mappings that the inverse image of every regularly open (resp. regularly closed) set is open (resp. closed) [2, Theorem 2.2]. A mapping is said to be *almost-open* if the image of every regularly open set is open.

3. Preliminaries. We begin by the following lemma.

Lemma 1. *If a mapping $f: X \to Y$ is a.c. and almost-open, then the inverse image $f^{-1}(V)$ of each regularly open set V of Y is a regularly open set of X.*

Proof. Let V be an arbitrary regularly open set of Y. Then, since f is *a.c.*, $f^{-1}(V)$ is open and hence we obtain that $f^{-1}(V) \subset \mathrm{Int}\,\mathrm{Cl}\,f^{-1}(V)$. In order to prove that $f^{-1}(V)$ is regularly open, it is sufficient to show that $f^{-1}(V) \supset \mathrm{Int}\,\mathrm{Cl}\,f^{-1}(V)$. Since f is *a.c.* and $\mathrm{Cl}\,V$ is regularly closed, $f^{-1}(\mathrm{Cl}\,V)$ is closed and hence we have $\mathrm{Int}\,\mathrm{Cl}\,f^{-1}(V) \subset \mathrm{Cl}\,f^{-1}(V) \subset f^{-1}(\mathrm{Cl}\,V)$. Since f is almost-open and $\mathrm{Int}\,\mathrm{Cl}\,f^{-1}(V)$ is regularly open, $f[\operatorname{Int}\operatorname{Cl} f^{-1}(V)]$ is open and hence we have $f[\operatorname{Int}\operatorname{Cl} f^{-1}(V)]\subset \operatorname{Int}\operatorname{Cl} V=V$. Therefore, we obtain that $f^{-1}(V) \supset \operatorname{Int}\operatorname{Cl} f^{-1}(V)$. Hence $f^{-1}(V)$ is a regularly open set in X.

Remark. The composition of *a.c.* mappings is not always *a.c.*, as the following counter-example shows.

Example. Let X be the set of all real numbers and $\Gamma_x=\{X,\phi\} \cup\{A\subset X \mid X-A:\text{ countable}\}$. We put $Y=\{a,b\}$, $\Gamma_y=\{Y,\{a\},\phi\}$, $Z=\{a,b,c\}$ and $\Gamma_z=\{Z,\{a,c\},\{a\},\{c\},\phi\}$. Consider a mapping $f:(X,\Gamma_x) \to(Y,\Gamma_y)$ defined as follows: $f(x)=a$ if x is rational; $f(x)=b$ if x is irrational and a mapping $g:(Y,\Gamma_y)\to(Z,\Gamma_z)$ defined as follows: $g(a)=a$ and $g(b)=b$. Then f is *a.c.* [2, Example 2.1]. Moreover, it is easy to check that g is continuous and hence *a.c.* But, by Example 2.3 of [2], $g\circ f$ is not *a.c.*

The above example shows that the composition of an *a.c.* mapping and a continuous mapping is not always *a.c.* While, we have the following lemma.

Lemma 2. *Let X, Y and Z be three spaces. If a mapping $f:X \to Y$ is a.c. and a mapping $g:Y\to Z$ is almost-open and a.c., then $g\circ f: X\to Z$ is a.c.*

Proof. Let W be an arbitrary regularly open set of Z. Then by Lemma 1 $g^{-1}(W)$ is regularly open in Y because g is almost-open and *a.c.* Since f is *a.c.*, $f^{-1}[g^{-1}(W)]=(g\circ f)^{-1}(W)$ is open in X. Hence $g\circ f$ is *a.c.*

4. Almost-continuous mappings and product spaces. Let $\{X_\alpha \mid \alpha\in I\}$ and $\{Y_\alpha \mid \alpha\in I\}$ be two families of spaces with the same set I of indices. We shall simply denote the product spaces $\Pi\{X_\alpha \mid \alpha\in I\}$ and $\Pi\{Y_\alpha \mid \alpha\in I\}$ by ΠX_α and ΠY_α respectively.

Theorem 1. *Let $f_\alpha: X_\alpha\to Y_\alpha$ be a mapping for each $\alpha\in I$ and $f:\Pi X_\alpha\to\Pi Y_\alpha$ a mapping defined by $f((x_\alpha))=(f_\alpha(x_\alpha))$ for each point (x_α) in ΠX_α. Then, f is a.c. if and only if f_α is a.c. for each $\alpha\in I$.*

Proof. For the sufficiency, see Theorem 2.10 of [2]. We shall prove the necessity. For each $\alpha\in I$, let $p_\alpha:\Pi X_\beta\to X_\alpha$ and $q_\alpha:\Pi Y_\beta\to Y_\alpha$ be the projections. Then, by the definition of f, we have $q_\alpha\circ f=f_\alpha\circ p_\alpha$ for each $\alpha\in I$. Since q_α is continuous open and f is *a.c.*, by Lemma 2, $q_\alpha\circ f$ is *a.c.* In order to prove that f_α is *a.c.*, we suppose that V_α is an arbitrary regularly open set in Y_α. Then $(f_\alpha\circ p_\alpha)^{-1}(V_\alpha)=(q_\alpha\circ f)^{-1}(V_\alpha)$ is open in ΠX_β. Since p_α is open and surjective, $p_\alpha[(f_\alpha\circ p_\alpha)^{-1}(V_\alpha)] =f_\alpha^{-1}(V_\alpha)$ is open in X_α. Hence f_α is *a.c.* for each $\alpha\in I$.

Theorem 2. *A mapping $h:X\to\Pi X_\beta$ is a.c. if and only if $p_\alpha\circ h$ is a.c. for each $\alpha\in I$, where p_α is the projection of ΠX_β onto X_α.*

Proof. *Necessity.* Suppose that h is *a.c.* Then, by Lemma 2, $p_\alpha\circ h$ is *a.c.* for each $\alpha\in I$ because p_α is open and continuous.

Sufficiency. Suppose that $p_\alpha \circ h$ is *a.c.* for each $\alpha \in I$. Let x be any point in X and V any neighborhood of $h(x)$ in ΠX_α. Then there exists an open set ΠV_α in ΠX_α such that $h(x) \in \Pi V_\alpha \subset V$, $V_\alpha = X_\alpha$ for all $\alpha \in I$ except a finite number of indices, say, $\alpha_1, \alpha_2, \cdots, \alpha_n$, and V_{α_i} is an open set in X_{α_i}, where $i=1,2,\cdots,n$. Since $p_\alpha \circ h$ is *a.c.* for each $\alpha \in I$, for each i there is a neighborhood U_{α_i} of x such that $(p_{\alpha_i} \circ h)(U_{\alpha_i}) \subset \text{Int Cl}\, V_{\alpha_i}$. Since we have $h(\bigcap_{i=1}^n U_{\alpha_i}) \subset \bigcap_{i=1}^n p_{\alpha_i}^{-1}[(p_{\alpha_i} \circ h)(U_{\alpha_i})] \subset \bigcap_{i=1}^n p_{\alpha_i}^{-1}[\text{Int Cl}\, V_{\alpha_i}]$, by Lemma 2 of [1], we obtain that $h(\bigcap_{i=1}^n U_{\alpha_i}) \subset \text{Int Cl}\, \Pi V_\alpha \subset \text{Int Cl}\, V$. Being $\bigcap_{i=1}^n U_{\alpha_i}$ a neighborhood of x, h is *a.c.*

References

[1] H. Kawashima: On the topological product of minimal Hausdorff spaces. TRU Math., **1**, 62–64 (1965).

[2] M. K. Singal and A. R. Singal: Almost-continuous mappings. Yokohama Math. J., **16**, 63–73 (1968).