42. *Symmetric Spaces Associated with Siegel Domains*

By Kazufumi NAKAJIMA
Department of Mathematics, Kyoto University

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Introduction. Let D be a Siegel domain of the second kind due to Pyatetski-Shapiro [2]. We then construct a symmetric Siegel domain in $\bar{D}$ which is invariant under a suitable equivalence. At the same time we establish a structure theorem of the Lie algebra of all infinitesimal automorphisms of the domain D.

1. Let $\mathfrak{g}=\sum_p \mathfrak{g}^p$ $(p\in \boldsymbol{Z}, [\mathfrak{g}^p, \mathfrak{g}^q]\subset \mathfrak{g}^{p+q})$ be a graded Lie algebra over $\boldsymbol{R}$ with $\dim \mathfrak{g}<\infty$. Then the radical $\mathfrak{r}$ of $\mathfrak{g}$ is a graded ideal. Concerning Levi decompositions of $\mathfrak{g}$, we obtain

Theorem 1. *There exists a semi-simple graded subalgebra* $\mathfrak{s}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g}=\mathfrak{s}+\mathfrak{r}$.

2. Denote by R (resp. by W) a real (resp. complex) vector space of a finite dimension, and by R_c the complexification of R. Let D be a Siegel domain of the second kind in $R_c\times W$ associated with a convex cone V in R and a V-hermitian form F on W. We denote by $\mathfrak{g}(D)$ the Lie algebra of all infinitesimal automorphisms of D. Kaup, Matsushima and Ochiai [1] showed that the Lie algebra $\mathfrak{g}(D)$ has the following graded structure:

$$\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2 \qquad ([\mathfrak{g}^p, \mathfrak{g}^q]\subset \mathfrak{g}^{p+q}),$$
$$\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^0 \qquad (\mathfrak{r}^p=\mathfrak{r}\cap \mathfrak{g}^p),$$

where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}(D)$. By using Theorem 1 we have

Theorem 2. *There exists a semi-simple graded subalgebra* $\mathfrak{s}=\sum_{p=-2}^{2}\mathfrak{s}^p$ *of* $\mathfrak{g}(D)$ *such that*

(1) $\mathfrak{s}^1=\mathfrak{s}^1$ *and* $\mathfrak{s}^2=\mathfrak{g}^2$,

(2) *For any* $X\in\mathfrak{s}^0$, *the condition* "$[X, \mathfrak{s}^1+\mathfrak{s}^2]=0$" *implies* $X=0$.

Let $\mathfrak{s}$ be as in Theorem 2. Since $\mathfrak{s}$ is semi-simple, there exists a unique element E_s of $\mathfrak{s}^0$ such that

$$[E_s, X]=pX \qquad \text{for } X\in\mathfrak{s}^p.$$

We set

$$\mathfrak{r}_0^{-2}=\{X\in r^{-2};\ [\mathfrak{s}, X]=0\},$$
$$\mathfrak{r}_s^{-2}=\{X\in r^{-2};\ [E_s, X]=-X\},$$
$$\mathfrak{r}_0^{0}=\{X\in r^{0};\ [\mathfrak{s}, X]=0\},$$
$$\mathfrak{r}_s^{0}=\{X\in r^{0};\ [E_s, X]=X\}.$$

In the notations as above, we have the following

Theorem 3. *The radical* $\mathfrak{r}$ *has the following structure*:

(1) $\mathfrak{r}^{-2}=\mathfrak{r}_0^{-2}+\mathfrak{r}_s^{-2}$ (*direct sum*), $\mathfrak{r}_0^{-2}\supset[\mathfrak{r}^{-1}, \mathfrak{r}^{-1}]$,

$\mathfrak{r}^0=\mathfrak{r}_0^0+\mathfrak{r}_s^0$ (*direct sum*).

(2) $\mathfrak{r}_s^{-2}=[\mathfrak{r}^{-2},\mathfrak{s}^0]=[\mathfrak{r}^0,\mathfrak{s}^{-2}]\supset[\mathfrak{r}^{-1},\mathfrak{s}^{-1}]$,
$\mathfrak{r}_s^0=[\mathfrak{r}^0,\mathfrak{s}^0]=[\mathfrak{r}^{-2},\mathfrak{s}^2]\supset[\mathfrak{r}^{-1},\mathfrak{s}^1]$,
$\dim \mathfrak{r}_s^{-2}=\dim \mathfrak{r}_s^0$.

(3) $\mathrm{ad}\, E_s=0$ *on* $\mathfrak{r}^{-1}$.

(4) $\mathfrak{r}_s^0$ *is an abelian ideal of* $\mathfrak{g}^0$ *satisfying the followings*:

a) $[\mathfrak{r}_s^0, \mathfrak{r}^{-1}+\mathfrak{r}_0^{-2}]=0$,

b) $[\mathfrak{r}_s^0, \mathfrak{r}_s^{-2}]\subset\mathfrak{r}_0^{-2}$.

3. Let $\mathfrak{s}$ be as in Theorem 2. Then we can see

$$(*)\qquad \begin{cases}\mathfrak{g}^{-2}=\mathfrak{s}^{-2}+\mathfrak{r}^{-2} \text{ (direct sum)},\\ \mathfrak{g}^{-1}=\mathfrak{s}^{-1}+\mathfrak{r}^{-1} \text{ (direct sum)}.\end{cases}$$

It is well known that the space $\mathfrak{g}^{-2}$ (resp. $\mathfrak{g}^{-1}$) can be identified with the space R (resp. W). Then the subspace $\mathfrak{s}^{-1}$ is a complex subspace. Denote by η_s the projection of $\mathfrak{g}_c^{-2}+\mathfrak{g}^{-1}$ $(=R_c\times W)$ onto $\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}$ corresponding to the decompositions $(*)$. And put $V_s=\eta_s(V)$. Then V_s is a convex cone in $\mathfrak{s}^{-2}$ and the restriction F_s of F to $\mathfrak{s}^{-1}$ is a V_s-hermitian form on $\mathfrak{s}^{-1}$. Let S be the Siegel domain of the second kind in $\mathfrak{s}_c^{-2}+\mathfrak{s}^{-1}$ associated with V_s and F_s.

Proposition 4. *The projection* η_s *maps* D *onto* S.

We can also prove

Theorem 5. *The Siegel domain* S *is a symmetric homogeneous domain and* $\mathfrak{s}$ *may be identified with* $\mathfrak{g}(S)$.

From the construction, we can see that S is contained in $\bar{D}$. Moreover we have

Proposition 6. *If* $\mathfrak{r}=0$, *then* $S=D$. *And if* $\mathfrak{r}\neq 0$, *then* S *is contained in the boundary of* D.

Proposition 4 gives a "*fibering*" of D. We have the following

Theorem 7. *Let* $a, b\in S$. *Then the fibers* $\eta_s^{-1}(a)$ *and* η_s^{-1} *are holomorphically equivalent to each other. Moreover every fiber is holomorphically equivalent to a bounded domain.*

The domain S is constructed from the subalgebra $\mathfrak{s}$. The following theorem implies the uniqueness of such domains.

Theorem 8. *Let* $\mathfrak{s}'$ *be another semi-simple graded subalgebra as in Theorem 2 and let* S' *be the corresponding symmetric domain. Then there exists* $X\in\mathfrak{g}^0$ *such that*

$$Ad\,(\exp X)\mathfrak{s}=\mathfrak{s}',\ \exp X(S)=S' \quad \textit{and} \quad \exp X\circ\eta_s=\eta_{s'}\circ\exp X.$$

Proof of Theorem 8 uses Theorem 3.

4. We now consider domains over classical cones. Denote by $H^+(m,\boldsymbol{R})$ (resp. by $H^+(m,\boldsymbol{C})$) the set of all positive definite real symmetric (resp. complex hermitian) matrices of degree m. And denote by $H^+(m,\boldsymbol{K})$ the set $\{X\in H^+(2m,\boldsymbol{C})\,;\, JX=\bar{X}J\}$, where

$$J=\begin{bmatrix} j & & & 0 \\ & j & & \\ & & \ddots & \\ 0 & & & j \end{bmatrix}, \qquad j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The sets $H^+(m, \boldsymbol{R})$, $H^+(m, \boldsymbol{C})$ and $H^+(m, \boldsymbol{K})$ are irreducible cones.

Proposition 9. *Let D be a Siegel domain over a cone stated above. Suppose* $\mathfrak{g}^{-2} \neq [\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. *Then* $\mathfrak{g}^1 = 0$.

Furthermore we can find the associated symmetric domain S for any homogeneous Siegel domain constructed in [2] over these cones. In particular we can calculate $\dim \mathfrak{g}^1$ and $\dim \mathfrak{g}^2$. The results are as follows.

(i) *The case* $V=H^+(m, \boldsymbol{R})$ $(m \geqq 2)$. Let $r(t)$ be an $\boldsymbol{N}$-valued non-decreasing function on the interval $[1, s]$ $(s \in \boldsymbol{N})$ such that $r(s) \leqq m$. Denote by $M(p, q, \boldsymbol{C})$ the vector space of all $p \times q$ complex matrices, and put $W=\{(u_{kt}) \in M(m, s, \boldsymbol{C})\,;\, u_{kt}=0 \text{ for } k>r(t)\}$. Define a V-hermitian form F on W by $F(u, v)=1/2(u\,{}^t\bar{v}+\bar{v}\,{}^tu)$.

Theorem 10. *Let D be a Siegel domain associated with V and F and let* $n=r(s)$. *Then* $\dim \mathfrak{g}^1=0$, $\dim \mathfrak{g}^2=1/2(m-n)(m-n+1)$ *and S is the Siegel domain of the first kind associated with the cone* $H^+(m-n, \boldsymbol{R})$.

(ii) *The case* $V=H^+(m, \boldsymbol{C})$ $(m \geqq 2)$. Let $r_h(t)$ be a function on $[1, s_h]$ as in (i) $(h=1, 2)$. And let W_h be the complex vector space corresponding to $r_h(t)$. We set $W=W_1 \times W_2$ and define a V-hermitian form F on W by $F(u, v)=u_1{}^t\bar{v}_1+\bar{v}_2{}^tu_2$, where $u=(u_1, u_2)$ and $v=(v_1, v_2)$. For the domain corresponding to V and F, we have

Theorem 11. *Assume* $r_1(s_1) \geqq r_2(s_2)$.

(1) *If* $r_2(s_2)=m$. *Then* $\dim \mathfrak{g}^1=\dim \mathfrak{g}^2=0$ *and* $S=(0)$.

(2) *If* $r_1(s_1)<m$. *Then* $\dim \mathfrak{g}^1=0$, $\dim \mathfrak{g}^2=(m-r_1(s_1))^2$ *and S is of the first kind associated with* $H^+(m-r_1(s_1), \boldsymbol{C})$.

(3) *If* $r_1(s_1)=m$ *and* $r_2(s_2)<m$. *Let* s_1' *be the integer* $(s_1'<s_1)$ *such that* $r_1(s_1')<r_1(s_1'+1)=m$. *(In the case* $r_1(1)=m$, *we put* $s_1'=r_1(s_1')=0$.*) And let* $n=\mathrm{Max}\,(r_1(s_1'), r_2(s_2))$. *Then* $\dim \mathfrak{g}^1=2(s_1-s_1')(m-n)$, $\dim \mathfrak{g}^2=(m-n)^2$ *and* $S=\{(z, w) \in M(m-n, m-n, \boldsymbol{C}) \times M(m-n, s_1-s_1', \boldsymbol{C})\,;\, \sqrt{-1}({}^t\bar{z}-z)-w\,{}^t\bar{w} \in H^+(m-n, \boldsymbol{C})\}$.

(iii) *The case* $V=H^+(m, \boldsymbol{K})$ $(m \geqq 2)$. Let $r(t)$ be an $\boldsymbol{N}$-valued non-decreasing function on $[1, s]$ such that $r(s) \leqq 2m$. And let $W=\{(u_{kt}) \in M(2m, s, \boldsymbol{C})\,;\, u_{kt}=0 \text{ for } k>r(t)\}$. Define a V-hermitian form F on W by $F(u, v)=1/2(u\,{}^t\bar{v}+J\bar{v}\,{}^tu\,{}^tJ)$.

Theorem 12. *Let D be the Siegel domain associated with V and F.*

(1) *If* $r(s)<2m-1$. *Let* $n=\left[\dfrac{r(s)+1}{2}\right]$. *Then* $\dim \mathfrak{g}^1=0$, $\dim \mathfrak{g}^2$

$=(m-n)(2m-2n-1)$ *and* S *is of the first kind associated with the cone* $H^+(m-n, \boldsymbol{K})$.

(2) *If* $r(s)=2m-1$. *Let* s' *be the integer* $(s'<s)$ *such that* $r(s')<2m-1$ *and* $r(s'+1)=2m-1$. (*In the case* $r(1)=2m-1$, *we put* $s'=0$.) *Then* $\dim \mathfrak{g}^1=2(s-s')$, $\dim \mathfrak{g}^2=1$ *and* $S=\{(z, w)\in \boldsymbol{C}^1\times M(1, s-s', \boldsymbol{C})\,;\ \mathrm{Im}\, z-w\,{}^t\overline{w}>0\}$.

(3) *If* $r(s-1)=2m$. *Then* $\dim \mathfrak{g}^1=\dim \mathfrak{g}^2=0$ *and* $S=(0)$.

(4) *If* $r(s)=2m$ *and* $r(s-1)<2m$. (*In the case* $s=1$, *we put* $r(0)=0$.) *Let* $n=\left[\dfrac{r(s-1)+1}{2}\right]$. *Then* $\dim \mathfrak{g}^1=4(m-n)$, $\dim \mathfrak{g}^2=(m-n)(2m-2n-1)$ *and* S *is the domain corresponding to the cone* $H^+(m-n, \boldsymbol{K})$ *and the function* $r(t)$ *such that* $s=1$ *and* $r(1)=2(m-n)$.

Remark. Proofs of Theorem 10, Theorem 11 and Theorem 12 partially use an idea due to T. Tsuji who also calculated $\dim \mathfrak{g}^1$ and $\dim \mathfrak{g}^2$ of Theorem 10, Theorem 11 and special cases in Theorem 12 by using different methods in his paper [3].

References

[1] W. Kaup, Y. Matsushima, and T. Ochiai: On the automorphisms and equivalences of generalized Siegel domains. Amer. J. Math., **92**, 475–497 (1970).

[2] I. I. Pyatetski-Shapiro: Geometry of Classical Domains and Theory of Automorphic Functions. Fizmatgiz, Moscow (1961) (French translation, Paris (1966)).

[3] T. Tsuji: Siegel domains over self-dual cones and their automorphisms (preprint).