40. *On the Existence of Global Solutions of Mixed Problem for Non-linear Boltzmann Equation*

By Seiji UKAI

Department of Applied Physics, Osaka City University

(Comm. by Kôsaku YOSIDA, M. J. A., March 12, 1974)

1. Introduction and summary. We study a mixed initial-boundary value problem of non-linear Boltzmann equation. The boundary condition considered here is the periodicity condition, to which the perfectly reflective boundary condition for the case of a rectangular domain can be reduced, [1]. Our hypotheses on collision operators are those for the so-called cut-off hard potentials, [2]. The solutions for the mixed problem have been known to exist only locally in time, [1]. Our aim is to show their global existence.

We denote by $f=f(t,x,\xi)$ the density distribution of gas particles at time $t\geqslant 0$ with respect to the position $x=(x_1,x_2,x_3)\in \boldsymbol{R}^3$ and velocity $\xi=(\xi_1,\xi_2,\xi_3)\in\boldsymbol{R}^3$. Our mixed problem is

(1.1a) $$\frac{\partial f}{\partial t}=-\sum_{i=1}^{3}\xi_i\frac{\partial f}{\partial x_i}+Q[f,f],$$

(1.1b) $$f \text{ is periodic in } x,$$

(1.1c) $$f\,|_{t=0}=f_0\geqslant 0,$$

(1.2) $$Q[f,g]=\frac{1}{2}\iint_{\boldsymbol{R}^3\times S^2} q(|\xi-\xi'|,\theta)\{f(\eta)g(\eta')+f(\eta')g(\eta)-f(\xi)g(\xi') -f(\xi')g(\xi)\}d\xi' d\omega.$$

Here $f(\eta)=f(t,x,\eta)$, etc., while $\omega\in S^2$, $\cos\theta=(\omega,\xi-\xi')/|\xi-\xi'|$, $\eta=\xi+(\omega,\xi-\xi')\omega$ and $\eta'=\xi'-(\omega,\xi-\xi')\omega$. The assumption of the cut-off hard potential means, [2], that there exist constants $q_0, q_1>0$, $0\leqslant\delta<1$ and for any $v>0$,

(1.3) $$0\leqslant q(v,\theta)\leqslant q_0|\cos\theta|(v+v^{-\delta}),\qquad \int_0^{\pi} q(v,\theta)\sin\theta d\theta\geqslant q_1 v(1+v)^{-1}.$$

Let $\Omega\subset\boldsymbol{R}^3$ be a fundamental rectangular domain of the periodicity condition. Let $g(\xi)=e^{-|\xi|^2}$ be a Maxwellian (Gaussian) distribution. With suitable changes of variables x, ξ and t, we may assume that, with $\{h_i(\xi)\}_{i=1}^5=\{1,\xi_1,\xi_2,\xi_3,|\xi|^2\}$,

(1.4) $$\iint_{\Omega\times\boldsymbol{R}^3} h_i(\xi)f_0(x,\xi)dxd\xi=\iint_{\Omega\times\boldsymbol{R}^3} h_i(\xi)g(\xi)dxd\xi,\qquad 1\leq i\leq 5.$$

Put $\psi_i=h_i g^{\frac{1}{2}}$. Define the (formal) operators L and Γ as

(1.5) $$Lu=g^{-\frac{1}{2}}Q[g^{\frac{1}{2}},g^{\frac{1}{2}}u],\qquad \Gamma[u,v]=g^{-\frac{1}{2}}Q[g^{\frac{1}{2}}u,g^{\frac{1}{2}}v].$$

Under the assumption (1.3), L takes the form, [2], (1.6) $$L=\Lambda+K;\ \Lambda u=-\nu(\xi)u,\qquad Ku=\int_{R^3}K(\xi,\xi')u(\xi')d\xi'.$$

Noting that $Q[g,g]=0$, [2], we put $f=g+g^{\frac{1}{2}}u$ and rewrite (1.1) with $u=u(t,x,\xi)$ as

(1.7a) $$\frac{\partial u}{\partial t}=-\sum_{i=1}^{3}\xi_i\frac{\partial u}{\partial x_i}+\Gamma[u,u],$$

(1.7b) $$u \text{ is periodic in } x,$$

(1.7c) $$\iint_{\Omega\times R^3}\psi_i u dx d\xi=0,\qquad t\geqslant 0,\ 1\leq i\leq 5,$$

(1.7d) $$u\,|_{t=0}=u_0.$$

Here the additional constraints (1.7c) arise from the well-known conservation laws, [1], for (1.1).

Let $l\geq 0$ and let $H^l(T^3)$ be the Sobolev space on the torus $T^3=\boldsymbol{R}^3/\Omega$. We may admit a non-integer for l. Let us consider the Hilbert space H_l of functions $u=u(x,\xi)$ defined as

(1.8) $$H_l=L^2_\xi(\boldsymbol{R}^3;H^l_x(T^3)),\qquad (u,v)_l=\int_{R^3}(u(\cdot,\xi),v(\cdot,\xi))_{H^l_x(T^3)}d\xi.$$

We define the operator B in H_l as

(1.9) $$B=-\sum_{i=1}^{3}\xi_i\frac{\partial}{\partial x_i}+L,$$

with $\mathcal{D}(B)=\{u\in H_l;\ \sum_{i=1}^{3}\xi_i u_{x_i}+\nu(\xi)u\in H_l\}$ as the domain of definition. The following is a key theorem for our existence proof.

Theorem 1.1. *Let $l\geqslant 0$, then,*

(i) *B is a generator of a contraction semi-group e^{tB}.*

(ii) *B has 0 as an eigenvalue with the five-dimensional eigenspace spanned by $\{\psi_i\}_{i=1}^{5}$. Denote its eigenprojection by P.*

(iii) $\exists\mu>0,\ \forall\gamma<\mu,\ \exists C_\gamma>0,\ \forall t\geqslant 0,\ \|e^{tB}(I-P)\|\leqslant C_\gamma e^{-\gamma t}$.

Once this theorem is proved, we can construct solutions of (1.7) by the iteration procedure by which Grad [1] obtained local solutions of (1.7) using only the estimate $\|e^{tB}\|\leqslant 1$.

More precisely, we shall consider the Banach space $H_{l,\beta}$;

(1.10) $$H_{l,\beta}=\{u(x,\xi);\ (1+|\xi|)^\beta u\in L^\infty_\xi(\boldsymbol{R}^3;H^l_x(\boldsymbol{T}^3)\},$$ $$\|u\|_{l,\beta}=\sup_{\xi\in R^3}(1+|\xi|)^\beta\,\|u(\cdot,\xi)\|_{H^l_x(T^3)},$$

and think of B of (1.9) as the operator in $H_{l,\beta}$ with $\mathcal{D}(B)=H_{l+1,\beta+1}$. We shall later see that $\Gamma[u,u]\in H_{l,\beta}$ whenever $u\in\mathcal{D}(B)$ if $l>0.5$, $\beta\geqslant 0$. We regard the mixed problem (1.7) as the evolution equation in $H_{l,\beta}$ with $l>0.5$, $\beta>1.5$;

(1.11) $$\frac{du(t)}{dt}=Bu(t)+\Gamma[u(t),u(t)],\quad u(t)\in \mathrm{Ker}\,(P),\quad u(+0)=u_0,$$

where du/dt is the derivative of u in the strong topology of $H_{l,\beta}$ and Ker (P) is the null space of the projection P on H_l defined in Theorem 1.1. Note that $H_{l,\beta}\subset H_l$ if $\beta>1.5$. Now, our existence theorem reads as

Theorem 1.2. *Let $l\geqslant 0.5$, $\beta\geqslant 1.5$, $\varepsilon>0$ and $0<\gamma<\mu$. Then there exist positive constants α_1 and α_2 such that, if $u_0\in H_{l+1+\varepsilon,\beta+1+\varepsilon}\cap \mathrm{Ker}\,(P)$ and if $\|u_0\|_{l+1+\varepsilon,\beta+1+\varepsilon}\leqslant\alpha_1$, then* (1.11) *has a unique solution*

$u(t)\in L_t^\infty([0,\infty);H_{l+1+\varepsilon,\beta+1+\varepsilon})\cap C_t^0([0,\infty);H_{l+1,\beta+1})\cap C_t^1([0,\infty);H_{l,\beta})$ *satisfying*

$$\|u(t)\|_{l+1+\varepsilon,\beta+1+\varepsilon}\leq\alpha_2 e^{-\gamma t}. \tag{1.12}$$

Thus the solution f of (1.1) tends to g exponentially when $t\to\infty$.

2. Lemmas on collision operators. We denote by $\sigma(A)$ the spectrum of an operator A, and by $\sigma_e(A)$, $\sigma_p(A)$, $\sigma_d(A)$ the essential, point, discrete spectrum of A.

The conditions (1.3) and that q is measurable lead to the following lemmas ([1], [2]),

Lemma 2.1. *$\nu(\xi)$ is measurable on $\boldsymbol{R}^3$ and $\nu_0\leq\nu(\xi)\leq\nu_1(1+|\xi|)$ with some positive constants ν_0,ν_1.*

Lemma 2.2. (i) *$K(\xi,\xi')$ is measurable and symmetric on $\boldsymbol{R}^3\times\boldsymbol{R}^3$.*

(ii) $\forall\beta\geq 0,\ \exists\kappa_\beta>0,\ \forall\xi\in\boldsymbol{R}^3,\ \int_{\boldsymbol{R}^3}|K(\xi,\xi')|(1+|\xi'|)^{-\beta}d\xi'\leq\kappa_\beta(1+|\xi|)^{-\beta-1}.$

(iii) $\exists\kappa>0,\ \forall\xi\in\boldsymbol{R}^3,\ \int_{\boldsymbol{R}^3}|\nu(\xi)^{-1}K(\xi,\xi')|^2\,d\xi'\leq\kappa.$

Lemma 2.3. (i) *L is self-adjoint, non-positive in H_l with $\mathscr{D}(L)=\mathscr{D}(\Lambda)=\{u\in H_l;\nu(\xi)u\in H_l\}$.*

(ii) *For L in $L_\xi^2(\boldsymbol{R}^3)$, $0\in\sigma_d(L)$ with the eigenspace spanned by $\{\psi_i\}_{i=1}^5$.*

Lemma 2.4. (i) *For any $l>1.5$ and $\beta\geqslant 1$, $\Lambda^{-1}\Gamma[u,v]$ is bounded from $H_{l,\beta}\times H_{l,\beta}$ into $H_{l,\beta}$; $\exists\eta_{l,\beta}>0,\ \forall u,v\in H_{l,\beta},\ \|\Lambda^{-1}\Gamma[u,v]\|\leqslant\eta_{l,\beta}\|u\|_{l,\beta}\|v\|_{l,\beta}$.*

(ii) *If $l>1.5$, $\beta>2.5$, and $u\in H_{l,\beta}$, then $\Gamma[u,u]\in\mathrm{Ker}\,(P)$. (Note that $\Gamma[u,u]=\Lambda\Lambda^{-1}\Gamma[u,u]\in H_{l,\beta-1}\subset H_l$ since $\beta-1>1.5$.)*

3. Outline of the proof of Theorem 1.1. In this section we think of Λ, K and L as operators in H_l. Define the operator A_0 as

$$\mathscr{D}(A_0)=\Big\{u\in H_l;\sum_{i=1}^3\xi_i u_{x_i}\in H_l\Big\},\qquad A_0=-\sum_{i=1}^3\xi_i\frac{\partial}{\partial x_i}. \tag{3.1}$$

It is anti-self-adjoint and generates a unitary group e^{tA_0}. Moreover, $\sigma(A_0)=i\boldsymbol{R}$ and $\sigma_p(A_0)=\{0\}$, corresponding eigenfunctions being constant in x. Since the C_0-semigroup $e^{t\Lambda}$ exists, $\|e^{t\Lambda}\|\leqslant e^{-\nu_0 t}$, and since A_0 and Λ commute, $e^{t\Lambda}e^{tA_0}$ form a C_0-semigroup, [3]. Denote its generator as A. It is not difficult to show

Lemma 3.1. (i) $A=A_0+\Lambda$, $\mathscr{D}(A)=\mathscr{D}(A_0)\cap\mathscr{D}(\Lambda)$. (ii) $\|e^{tA}\|\leqslant e^{-\nu_0 t}$. (iii) $\sigma(A)=\sigma_e(A)=\{\lambda;\mathrm{Re}\,\lambda=-\nu(\xi),\xi\in\boldsymbol{R}^3\}$.

Thus the resolvent $(\lambda-A)^{-1}$ exists if $\mathrm{Re}\,\lambda>-\nu_0$. The following is a key to the proof of Theorem 1.1.

Lemma 3.2. *Let $l\geqslant 0$.* (i) *$K(\lambda-A)^{-1}$ is compact on H_l, $\mathrm{Re}\,\lambda>\nu_0$.* (ii) *For any $\beta>-\nu_0$, $\|K(\lambda-A)^{-1}\|\to 0$ ($|\lambda|\to\infty$) uniformly for λ, $\mathrm{Re}\,\lambda\geqslant\beta$.*

The proof is carried out with the aid of the Fourier series representation of $u \in H_l$ and

Lemma 3.3. *$A^{-1}K$ and consequently KA^{-1} is compact on $L^2_\xi(\mathbf{R}^3)$.*

Let us define the operator B as

$$B = A + K = A_0 + L, \qquad \mathcal{D}(B) = \mathcal{D}(A) = \mathcal{D}(A_0) \cap \mathcal{D}(L). \tag{3.2}$$

Since K is bounded, B is a generator of C_0-semigroup e^{tB}, and owing to Lemma 2.3, B is dissipative. Therefore we see

Theorem 3.1. *B is maximally dissipative.*

As a consequence, $\sigma(B) \subset \{\lambda ; \operatorname{Re}\lambda \leq 0\}$. On the other hand Lemma 3.2, (i) implies that K is A-compact, [4], so that $\sigma_e(B) = \sigma_e(A)$ and each $\lambda \in \sigma_d(B)$ is a pole of the resolvent $(\lambda - B)^{-1}$. Moreover,

Theorem 3.2. (i) *For any $\beta > -\nu_0$, $\sigma_d(B) \cap \{\lambda ; \operatorname{Re}\lambda \geqslant \beta\}$ is a finite set.* (ii) *$^\exists\mu > 0$, $\sigma_d(B) \setminus \{0\} \subset \{\lambda ; \operatorname{Re}\lambda \leq -\mu\}$.* (iii) *$0 \in \sigma_d(B)$ with the eigenspace spanned by $\{\psi_i\}_{i=1}^5$, and is a simple pole of $(\lambda - B)^{-1}$.*

Proof. (i) follows from Lemma 3.2, (ii). Suppose $\gamma \in \mathbf{R}^1$, $i\gamma \in \sigma_d(B)$, and let u be a corresponding eigenfunction; $Bu = i\gamma u$. Then $(A_0 u, u)_l + (Lu, u)_l = i\gamma(u, u)_l$ whence $Lu = 0$ by the non-positivity of L and $A_0 u = i\gamma u$, i.e., $i\gamma \in \sigma_p(A_0) = \{0\}$. This and (i) imply (ii).

The Fourier series representation gives

Lemma 3.4. *Write $\lambda = \beta + i\gamma$. For any $\beta > -\nu_0$, $u \in H_l$,*

$$\begin{aligned} &\int_{-\infty}^{\infty} \|(\lambda - A)^{-1}u\|_l^2 \, d\gamma \leqslant \frac{\pi}{\beta + \nu_0} \|u\|_l^2, \\ &\int_{-\infty}^{\infty} \|(\lambda - A^*)^{-1}u\|_l^2 \, d\gamma \leqslant \frac{\pi}{\beta + \nu_0} \|u\|_l^2. \end{aligned} \tag{3.3}$$

Since $\mathcal{D}(B) = \mathcal{D}(A)$, the second resolvent equation holds, giving

$$\begin{aligned} &(\lambda - B)^{-1} = (\lambda - A)^{-1} + \tilde{Z}(\lambda); \\ &\tilde{Z}(\lambda) = (\lambda - A)^{-1}(I - K(\lambda - A)^{-1})^{-1}K(\lambda - A)^{-1}, \end{aligned} \tag{3.4}$$

and, since both e^{tB} and e^{tA} are C_0-semigroups, we have, by the inverse Laplace transformation, for any $\beta_0 > 0$,

$$e^{tB}u = e^{tA}u + s - \lim_{a \to \infty} \frac{1}{2\pi i} \int_{\beta_0 - ia}^{\beta_0 + ia} e^{\lambda t} \tilde{Z}(\lambda) u \, d\lambda, \qquad t > 0, \; u \in \mathcal{D}(B). \tag{3.5}$$

Define the integral

$$Z_\beta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\gamma t} \tilde{Z}(\beta + i\gamma) d\gamma. \tag{3.6}$$

Lemmas 3.2 and 3.4 lead to

Lemma 3.5. *Let $\beta > -\nu_0$ be such that $\lambda \notin \sigma_d(B)$ for all λ,* $\operatorname{Re}\lambda = \beta$.

(i) *The integral $Z_\beta(t)$ converges absolutely in the weak operator topology of H_l.* (ii) $^\exists C_\beta > 0$, $^\forall t \geqslant 0$, $\|Z_\beta(t)\| \leqslant C_\beta$.

Theorem 3.3. *Let $-\mu < \beta < 0$, then*

$$e^{tB} = e^{tA} + e^{\beta t} Z_\beta(t) + P, \qquad t \geqslant 0. \tag{3.7}$$

Proof. Lemma 3.2, (ii) allows us to shift the integration path in (3.5) to the line $\operatorname{Re}\lambda = \beta$, and $\operatorname{Res}_{\lambda=0} e^{\lambda t}\tilde{Z}(\lambda) = \operatorname{Res}_{\lambda=0} e^{\lambda t}(\lambda - B)^{-1} = P$ by Theorem 3.2, (iii).

Now, Theorem 1.1 follows from Theorems 3.1, 3.2, 3.3 and Lemma 3.5, (ii).

4. Outline of the proof of Theorem 1.2. Let $\varepsilon>0$. Define
$S_{l,\beta,\varepsilon}=L_t^\infty([0,\infty);H_{l+1+\varepsilon,\beta+1+\varepsilon})\cap C_t^0([0,\infty);H_{l+1,\beta+1})\cap C_t^1([0,\infty);H_{l,\beta})$, and put, with $\gamma\in\boldsymbol{R}^1$,

(4.1) $$\|u\|=\|u\|_{l+1+\varepsilon,\beta+1+\varepsilon},\qquad |u|_\gamma=\sup_{t\geqslant 0} e^{\gamma t}\|u(t)\|.$$

In this section we think of operators Λ, K, A and B in $H_{l,\beta}$ with $\mathscr{D}(\Lambda)=\mathscr{D}(A)=\mathscr{D}(B)=H_{l+1,\beta+1}$. The operators e^{tA} of the previous section form a semigroup also on $H_{l,\beta}$, though it is neither strongly continuous nor has A as a generator. Nevertheless, we can show

Lemma 4.1. *Let $u_0\in H_{l+1+\varepsilon,\beta+1+\varepsilon}$, $\varepsilon>0$, and put $\varphi(t)=e^{tA}u_0$.*

(i) $\varphi(t)\in S_{l,\beta,\varepsilon}$. (ii) $\|\varphi(t)\|\leqslant e^{-\nu_0 t}\|u_0\|$. (iii) $\varphi'(t)=A\varphi(t)$ *where $\varphi'(t)$ is the derivative of $\varphi(t)$ in the strong topology of $H_{l,\beta}$.*

The following lemma plays an essential role for our purpose, the assertion (ii) of which is due to Grad [1].

Lemma 4.2. *Let $f(t)\in S_{l,\beta,\varepsilon}$ and define $\chi(t)=\int_0^t e^{(t-s)A}\Lambda f(s)ds$.*

(i) $\chi(t)\in S_{l,\beta,\varepsilon}$. (ii) $\forall\gamma<\nu_0$, $|\chi|_\gamma\leqslant\max\left(1,\dfrac{\nu_0}{\nu_0-\gamma}\right)|f|_\gamma$.

(iii) $\chi'(t)=A\chi(t)+\Lambda f(t)$, *in $H_{l,\beta}$.*

Put $w(t)=\varphi(t)+\chi(t)$ and consider the integral equation

(4.3) $$u(t)=w(t)+\int_0^t e^{(t-s)A}Ku(s)ds.$$

Using the Neumann series, we easily see

Lemma 4.3. *Let u_0 and $f(t)$ be as in Lemmas* 4.1 *and* 4.2. *Then,* (4.3) *has a unique solution $u(t)\in S_{l,\beta,\varepsilon}$, which is also a unique solution within $S_{l,\beta,\varepsilon}$ of the equation*

(4.4) $$u'(t)=Bu(t)+\Lambda f(t),\qquad in\ H_{l,\beta};\ u(+0)=u_0.$$

By virtue of Theorem 1.1, (iii), we can show, using the arguments given in [1],

Lemma 4.4. *Let $l\geqslant 0$, $\beta\geqslant 1.5$, $\gamma<\nu_0$, and let u_0, $f(t)$ and $u(t)$ be those of the previous lemma. Suppose, further, that $u_0, \Lambda f(t)\in$* Ker (P), *then, with some positive constants $a=a_{l,\beta,\gamma,\varepsilon}$ and $b=b_{l,\beta,\gamma,\varepsilon}$, we have $u(t)\in$* Ker (P) *and*

(4.5) $$|u|_\gamma\leqslant a\|u_0\|+b|f|_\gamma.$$

Proof of Theorem 1.2. Let $l\geqslant 0.5$, $\beta\geqslant 1.5$, $\varepsilon>0$ and let $u_0\in H_{l+1+\varepsilon,\beta+1+\varepsilon}\cap$ Ker (P). Consider the integral equation

(4.6) $$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}Ku(s)ds+\int_0^t e^{(t-s)A}\Gamma[u(s),u(s)]ds,$$

which we solve by the iteration

(4.7) $$u^n(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}Ku^n(s)ds+\int_0^t e^{(t-s)A}\Gamma[u^{n-1}(s),u^{n-1}(s)]ds,$$

for $n \geqslant 1$ aud $u^0(t)=0$. Lemmas 2.4 and 4.4 imply $u^n(t) \in S_{l,\beta,\varepsilon} \cap \operatorname{Ker}(P)$ and for $0 \leq \gamma < \nu_0$, with $\eta = \eta_{l+1+\varepsilon, \beta+1+\delta}$,

$$|u^n|_\gamma \leqslant a \|u_0\| + b\eta |u^{n-1}|_\gamma^2, \tag{4.8}$$

Choose α_1 and α_2 as

$$0 < \alpha_1 < \frac{1}{4ab\eta}, \qquad \alpha_2 = \frac{1}{2b\eta}(1 - \sqrt{1 - 4ab\eta\alpha_1}). \tag{4.9}$$

Let $\|u_0\| \leqslant \alpha_1$, then $|u^n|_\gamma \leqslant \alpha_2$, from (4.8). Similarly, we can get

$$|u^{n+1} - u^n|_\gamma \leqslant 2b\eta\alpha_2 |u^n - u^{n-1}|_\gamma, \qquad n \geqslant 1 \tag{4.10}$$

whence $u^n(t)$ converges in $H_{l+1+\varepsilon, \beta+1+\varepsilon}$ to some $u(t) \in L_t^\infty([0,\infty); H_{l+1+\varepsilon,\beta+1+\varepsilon}) \cap C_t^0([0,\infty); H_{l+1,\beta+1}) \cap \operatorname{Ker}(P)$ uniformly in t, and $|u|_\gamma \leqslant \alpha_2$. It is not difficult to see that this $u(t)$ is a desired solution of (1.11).

A more detailed exposition will be presented elsewhere.

Acknowledgement. The author would like to express his gratitude to Professor Y. Shizuta and Professor K. Asano for their valuable suggestions and to Professor M. Yamaguchi for his continuous encouragement.

References

[1] Grad, H.: Proc. Symp. in Appl. Math., **17**, 154–183 (1965).
[2] ——: Rarefied Gas Dynamics, I, 26–59 (1963).
[3] Trotter, H. F.: Proc. Am. Math. Soc., **10**, 545–551 (1959).
[4] Kato, T.: Perturbation Theory of Linear Operators. Springer (1966).