83. *Spherical Matrix Functions on Locally Compact Groups*

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Introduction. Let G be a locally compact and σ-compact unimodular group, and K a compact subgroup of G. Let Ω_K be the set of all equivalence classes of irreducible representations of K. For every $\delta \in \Omega_K$, χ_δ denotes the trace of δ.

In 1952, R. Godement defined spherical functions in [1]. Let $\{\mathfrak{H}, T_x\}$ be a completely irreducible representation of G on a Banach space $\mathfrak{H}$. If the subspace $\mathfrak{H}(\delta)=E(\delta)\mathfrak{H}$, where

$$E(\delta)=\int_K T_k\bar{\chi}_\delta(k)dk,$$

is of pd-dimensional ($d=$degree of δ), he called the function

$$\phi_\delta(x)=\mathrm{Tr}\,[E(\delta)T_x]$$

a spherical function of type δ of height p.

In this paper, we shall consider spherical matrix functions instead of spherical functions. Here, a spherical matrix function means a matrix-valued continuous function $U=U(x)$ on G such that

(1) $\chi_\delta * U=U$,

(2) $\int_K U(kxk^{-1}y)dk=U(x)U(y)$,

and

(3) $\{U(x)\,;\, x\in G\}$ is an irreducible family of matrices.

Using Theorem in [4], we see that the function

$$\phi(x)=d\cdot \mathrm{Tr}\,[U(x)]$$

is a spherical function, and conversely, every spherical function is given in this form. By considering spherical matrix functions, Theorems 10, 14 in [1] on spherical functions of height one can be generalized for arbitrary spherical matrix functions (Theorems 1, 3 respectively). And if there exists a closed subgroup P of G such that $G=KP$ and $K\cap P=\{e\}$, we can give an example of matrix functions which satisfy the conditions (1) and (2). Especially, if G is a connected semi-simple Lie group with finite center and K a maximal compact subgroup of G, we obtain a generalization of a well known formula which gives spherical functions of type 1 (Theorem 4).

In § 4, we shall mention that under what conditions a topologically irreducible representation becomes quasi-simple in the sense of Harish-

Chandra.

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§ 1. Spherical matrix functions. Let G be a locally compact and σ-compact unimodular group, and K a non-trivial compact subgroup of G. For every $\delta\in\Omega_K$, we shall denote by $L^\circ(\delta)$ the algebra of all continuous functions f on G with compact supports such that $\bar{\chi}_\delta * f = f * \bar{\chi}_\delta = f$ and that

$$f(x)=f^\circ(x)=\int_K f(kxk^{-1})dk \qquad \text{for all } x\in G.$$

Of course, the product in $L^\circ(\delta)$ is convolution, and $\bar{\chi}_\delta * f$ means the convolution of $\bar{\chi}_\delta$ and f.

Let $\{\mathfrak{H}, T_x\}$ be a representation of G. Hereafter, the representation space $\mathfrak{H}$ will be always a Hausdorff, complete, locally convex topological vector space. Let $\mathfrak{H}(\delta)$ be the set of vectors in $\mathfrak{H}$ which are transformed according to δ under $k\to T_k$, and

$$E(\delta)=\int_K T_k\bar{\chi}_\delta(k)dk$$

a usual projection from $\mathfrak{H}$ onto $\mathfrak{H}(\delta)$. If the representation $\{\mathfrak{H}, T_x\}$ is topologically irreducible and $\dim \mathfrak{H}(\delta)=pd<+\infty$ (where d is the degree of δ), the continuous function

$$\phi_\delta(x)=\mathrm{Tr}\,[E(\delta)T_x]$$

on G is called a spherical function of type δ of height p [3]. These spherical functions were defined by R. Godement for completely irreducible representations on Banach spaces [1].

If a representation $\{\mathfrak{H}, T_x\}$ of G satisfies $\dim\mathfrak{H}(\delta)=pd<+\infty$, the restriction $\tilde{T}_k$ of T_k on $\mathfrak{H}(\delta)$ is a pd-dimensional representation of K on $\mathfrak{H}(\delta)$. Moreover, we can take a base $v_1,\cdots,v_{pd}$ in $\mathfrak{H}(\delta)$ such that $\tilde{T}_k$ is written in matrix form

$$\tilde{T}_k=\begin{pmatrix} D(k) & & 0\\ & \ddots & \\ 0 & & D(k)\end{pmatrix}$$

with respect to this base, where $D(k)$ is an irreducible unitary representation of K belonging to δ. Let $v'_1,\cdots,v'_{pd}$ be continuous linear functionals on $\mathfrak{H}$ such that

$$(v_i, v'_j)=\delta_{ij} \qquad (1\leqq i,\ j\leqq pd),$$

and put

$$u_{ij}(x)=d^{-1}\sum_{\nu=1}^{d}(E(\delta)T_x v_{(j-1)d+\nu},\ v'_{(i-1)d+\nu}).$$

Then the matrix function $U_\delta(x)=(u_{ij}(x))_{1\leqq i,j\leqq p}$ on G satisfies

(1) $\chi_\delta * U_\delta = U_\delta$,

(2) $\displaystyle\int_K U_\delta(kxk^{-1}y)dk=U_\delta(x)U_\delta(y) \qquad$ for all $x, y\in G$,

and

$$f \to U_\delta(f) = \int_G U_\delta(x) f(x) dx$$

is a p-dimensional representation of the algebra $L^\circ(\delta)$. If the representation $\{\mathfrak{H}, T_x\}$ is topologically irreducible,

(3) U_δ is irreducible,

i.e., $\{U_\delta(x); x \in G\}$ is an irreducible family of matrices, and the representation $f \to U_\delta(f)$ of $L^\circ(\delta)$ is irreducible. Moreover, we obtain the equality

$$\phi_\delta(x) = d \cdot \mathrm{Tr}\,[U_\delta(x)].$$

Definition. If a continuous matrix function $U = U(x)$ satisfies the above conditions (1), (2) and (3), it is called a spherical matrix function of type δ.

Two spherical matrix functions $U = U(x)$ and $V = V(x)$ are called equivalent if there exists a regular matrix S such that $U(x) = S^{-1}V(x)S$ for all $x \in G$. Then, using Theorem in [4], we can prove the following

Theorem 1. *For every $\delta \in \Omega_K$, the relation*

$$\phi(x) = d \cdot \mathrm{Tr}\,[U(x)]$$

gives an explicit one to one correspondence between the set of all spherical functions of type δ and that of all equivalence classes of spherical matrix functions of the same type.

It was proved by R. Godement [1] that spherical functions of type δ of height one satisfy the conditions (1) and (2). The above theorem is a generalization of this result by R. Godement.

§ 2. Spherical matrix functions on connected Lie groups. Let G be a connected unimodular Lie group, and K a nontrivial compact analytic subgroup of G. Let $U(G)$ be the algebra of all distributions on G whose carriers reduce to the identity. When a representation $\{\mathfrak{H}, T_x\}$ of G is K-finite, i.e., $\dim \mathfrak{H}(\delta) < +\infty$ for all $\delta \in \Omega_K$, we put

$$\mathfrak{H}_K = \sum_{\delta \in \Omega_K} \mathfrak{H}(\delta).$$

As is well known, a representation π_K of the algebra $U(G)$ is defined on $\mathfrak{H}_K$ [1], and π_K is algebraically irreducible if $\{\mathfrak{H}, T_x\}$ is topologically irreducible [1].

Theorem 2. *Let $\{\mathfrak{H}^1, T_x^1\}$ and $\{\mathfrak{H}^2, T_x^2\}$ be two K-finite topologically irreducible representations of G. Then the following three statements are equivalent.*

(i) *There exists at least one $\delta \in \Omega_K$ such that $\phi_\delta^1 = \phi_\delta^2 \neq 0$ where ϕ_δ^i $(i=1,2)$ are spherical functions of type δ defined by $\{\mathfrak{H}^i, T_x^i\}$.*

(ii) *For all $\delta \in \Omega_K$, $\phi_\delta^1 = \phi_\delta^2$.*

(iii) *The corresponding algebraically irreducible representations π_K^1 and π_K^2 of $U(G)$ are equivalent.*

The analogous theorem holds also for spherical matrix functions.

For $\alpha \in U(G)$ we define a distribution α° by $\alpha^\circ(f) = \alpha(f^\circ)$, and denote by $U^\circ(G)$ the algebra of all distributions $\alpha \in U(G)$ such that $\alpha = \alpha^\circ$. And also we define a distribution α' by $\alpha'(f) = \alpha(f')$ where $f'(x) = f(x^{-1})$.

Then we can prove the following theorem which is a generalization of Theorem 14 in [1] for spherical functions of height one.

Theorem 3. *Let $U = U(x)$ be an irreducible continuous matrix function on G. Then we have*

$$\int_K U(kxk^{-1}y)dk = U(x)U(y)$$

for all $x, y \in G$ if and only if (U is analytic and) the equation

$$\alpha' * U = U(\alpha)U$$

is satisfied for every $\alpha \in U^\circ(G)$.

§ 3. An example of spherical matrix functions. Let G be a locally compact and σ-compact unimodular group, and K a non-trivial compact subgroup of G. We assume that there exists a closed subgroup P of G such that

$$G = KP, \qquad K \cap P = \{e\},$$

and that the decomposition $x = kp$ ($k \in K, p \in P$) is continuous. We shall denote by $D(k)$ an irreducible unitary representation of K belonging to $\delta \in \Omega_K$, and by $\Lambda(p)$ a finite-dimensional irreducible representation of P. Then the matrix function

$$U_{\Lambda,\delta}(x) = \int_K V_{\Lambda,\delta}(kx^{-1}k^{-1})dk,$$

where $V_{\Lambda,\delta}(x) = \Lambda(p^{-1}) \otimes \overline{D(k)}$ ($x = kp$), satisfies the conditions (1) and (2) in § 1. This is just the matrix function defined by the representation of G induced from Λ.

Now let's assume that G is a connected semi-simple Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$, where, as usual, $\mathfrak{k}$ denotes a maximal compact subalgebra. Let $\mathfrak{h}^-$ be a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{h}^- + \mathfrak{n}$ an Iwasawa decomposition of $\mathfrak{g}$, and $G = KAN$ the corresponding Iwasawa decomposition of G. Since every finite-dimensional irreducible representation of $P = AN$ is one-dimensional, it is considered as a one-dimensional representation of A. Therefore, for every one-dimensional representation λ of A, a matrix function $U_{\lambda,\delta}$ is defined as above.

In general, if a matrix function $U = U(x)$ satisfies the conditions (1) and (2) in § 1, we can find a regular matrix S such that

$$S^{-1}U(x)S = \begin{pmatrix} U^1(x) & & * \\ & \ddots & \\ 0 & & U^r(x) \end{pmatrix},$$

where $U^i(x)$ ($i = 1, \cdots, r$) are spherical matrix functions. Then we call $U^i(x)$ the irreducible components of U. Now, using Theorem 5.5.1.5 in [5] and Theorem 2 in this paper, we can prove

Theorem 4. *Let G be a connected semi-simple Lie group with finite center, and* $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ *a Cartan decomposition of the Lie algebra* $\mathfrak{g}$ *of G, where* $\mathfrak{k}$ *denotes a maximal compact subalgebra. Let* $\mathfrak{h}^{-}$ *be a maximal abelian subalgebra of* $\mathfrak{p}, \mathfrak{g}=\mathfrak{k}+\mathfrak{h}^{-}+\mathfrak{n}$ *an Iwasawa decomposition of* $\mathfrak{g}$*, and $G=KAN$ the corresponding Iwasawa decomposition of G. Then, for every spherical matrix function U of type δ ($\delta \in \Omega_K$), there exists a one-dimensional representation λ of A such that U is equivalent to an irreducible component of* $U_{\lambda,\delta}$.

§ 4. Topologically irreducible nice representations. Let $\{\mathfrak{H}, T_x\}$ be a representation of a locally compact unimodular group G. Let's say that it is nice if there exists a non-trivial compact subgroup K' of G such that

$$0<\dim \mathfrak{H}(\delta')<+\infty$$

for some $\delta' \in \Omega_K$.

Theorem 5. *Let G be a locally compact and σ-compact unimodular group, and K a non-trivial compact subgroup of G. Then, if $L^{\circ}(\delta)$ has sufficiently many irreducible representations whose dimensions are $\leqq p$, δ is contained at most p-times in every completely irreducible representation and in every topologically irreducible nice representation of G.*

If G is a connected semi-simple Lie group with finite center and K a maximal compact subgroup of G, it is shown, using Corollary 5.5.1.8. in [5], that $L^{\circ}(\delta)$ has sufficiently many irreducible representations whose dimensions are $\leqq d=$ degree of δ. Thus a topologically irreducible representation of G is nice if and only if it is K-finite. From this fact, we obtain the following

Theorem 6. *Let G be a connected semi-simple Lie group with finite center. If a topologically irreducible representation $\{\mathfrak{H}, T_x\}$ of G is nice, then*

(i) *T_ζ is a scalar multiple of the identity operator on the Gårding subspace of $\mathfrak{H}$ for all ζ in the center of $U(G)$,*

(ii) *T_z is a scalar multiple of the identity operator on $\mathfrak{H}$ for all z in the center of G.*

This theorem gives a characterization of quasi-simple irreducible representations in the sense of Harish-Chandra [2]: a topologically irreducible representation of G is quasi-simple if it is nice.

References

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