82. The Connection between the Order and the Diameter of a Neighborhood in a Vector Space

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In this paper, we investigate the connection between the order and the convergence exponent of the diameter of a bounded set in a normed space. We apply then the obtained results to a locally convex topological vector space.

1. Let E be a vector space over the field of real or complex numbers and A and B arbitrary sets in E.

For each positive number ε, let $M(A,B;\varepsilon)$ be the supremum of all natural numbers m, for which there exist elements $x_1,\cdots,x_m\in A$ with $x_i-x_j\notin\varepsilon B$ for $i\neq j$ $(1\leqq i,j\leqq m)$. Let $\rho(A,B)$ be the infimum of all positive numbers ρ, for which there is a positive number ε_0 such that $M(A,B;\varepsilon)<\exp(\varepsilon^{-\rho})$ for $0<\varepsilon<\varepsilon_0$. If no number ρ with the given property exists we set $\rho(A,B)=+\infty$. We then call $\rho(A,B)$ the *order of A with respect to B*; as is easily seen, we have

$$\rho(A,B)=\overline{\lim_{\varepsilon\to 0}}\{\log\log M(A,B;\varepsilon)/\log\varepsilon^{-1}\}.$$

The infimum $\delta_n(A,B)$ of all positive numbers δ, for which there is a vector subspace F of E of dimension at most n with $V\subset\delta U+F$ is called the *n-th diameter of A with respect to B.*

Let $a_1,a_2,\cdots$ be a sequence of positive numbers converging to zero. We call the infimum λ, of those values μ for which the series $\sum_{n=1}^{\infty}a_n^{\mu}$ converges, the *exponent of convergence* of the sequence $\{1/a_n\}$, and we call the exponent of convergence of the sequence $\{\log a_n^{-1}\}$ the *convergence type* of the sequence $\{a_n\}$. Let ε be a positive number, then we have the following two lemmas.

Lemma 1. *Let λ be the exponent of convergence of the sequence $\{1/a_n\}$. Then $\lambda=\overline{\lim}_{\varepsilon\to 0}\{\log m(\varepsilon)/\log\varepsilon^{-1}\}$, where $m(\varepsilon)$ denotes the number of terms of the sequence $\{a_n\}$ which are greater than ε.*

For a proof see [1], p. 89.

Lemma 2. *Let τ be the convergence type of the sequence $\{a_n\}$. Then*

$$\tau=\overline{\lim_{\varepsilon\to 0}}\{\log m(\varepsilon)/\log\log\varepsilon^{-1}\}.$$

Proof. Applying Lemma 1 to the sequence $\{\log a_n^{-1}\}$, we see that $\tau=\overline{\lim}_{\delta\to 0}\{\log l(\delta)/\log\delta^{-1}\}$ $(\delta>0)$, where $l(\delta)$ is the number of terms of $\{\log a_n^{-1}\}$ greater than δ. But obviously $l(\delta)=m(e^{-1/\delta})$. Therefore

$$\tau=\overline{\lim_{\delta\to 0}}\{\log m(e^{-1/\delta})/\log \delta^{-1}\}.$$

Replacing $e^{-1/\delta}$ by ε, we obtain the lemma.

Let E be a real normed space and U the closed unit ball of E. Then we have the following lemmas.

Lemma 3. *For each bounded subset B of E, $\delta_n(B, U)\leqq\varepsilon/4$ implies the inequality*

$$M(B, U;\varepsilon)\leqq(4\delta_0(B, U)\varepsilon^{-1}+2)^n.$$

This is shown by a modification of Lemma 1 (p. 144) of [4].

Lemma 4. *For each absolutely convex bounded subset B of E, the inequality*

$$\delta_0(B, U)\cdots\delta_n(B, U)\leqq(n+1)!\,\varepsilon^{n+1}M(B, U;\varepsilon)$$

is valid for all non-negative integers n and $\varepsilon>0$.

For a proof see [4], p. 145.

Lemma 5. *Let B be an absolutely convex bounded subset of E, and let λ be the exponent of convergence of the sequence $\{\delta_n(B, U)^{-1}\}$. Then $\rho(B, U)\leqq\lambda$. If $\rho(B, U)<1$, then*

$$\lambda\leqq\rho(B, U)/\{1-\rho(B, U)\}.$$

Proof. First, for any $\varepsilon>0$, let $m(\varepsilon)$ be the number of terms of the sequence $\{\delta_n(B, U)\}$ which are greater than ε. Since $m(\varepsilon/4)=n$ implies $\delta_n(B, U)\leqq\varepsilon/4$, we have

$$M(B, U;\varepsilon)\leqq(4\delta_0(B, U)\varepsilon^{-1}+2)^n$$

by Lemma 3. Therefore

$$\{\log\log M(B, U;\varepsilon)\}/\log\varepsilon^{-1}\leqq\log m(\varepsilon/4)/\log\varepsilon^{-1}+\gamma(\varepsilon),$$

where $\gamma(\varepsilon)=\{\log\log 6\delta_0\varepsilon^{-1}\}/\log\varepsilon^{-1}$. But, then since $\lim_{\varepsilon\to 0}\gamma(\varepsilon)=0$, we obtain $\rho(B, U)\leqq\lambda$ by Lemma 1.

Next, let $\rho(B, U)<1$, then for any ρ' with $\rho(B, U)<\rho'<1$, there exist $\varepsilon_0>0$ and ρ such that $\rho(B, U)\leqq\rho<\rho'$ and $M(B, U;\varepsilon)\leqq\exp(\varepsilon^{-\rho})$ for all ε with $0<\varepsilon<\varepsilon_0$. Put $\mu=\rho/\{1-\rho\}$. If n_0 is an integer with $(n_0+1)^{(1/\mu+1)}\varepsilon_0>1$ then

$$\delta_n(B, U)\leqq e(n+1)^{-1/\mu}\qquad\text{for all } n\geqq n_0.$$

In fact, if $\delta_m(B, U)>e(m+1)^{-1/\mu}$ for some integer $m\geqq n_0$ we obtain the inequality

$$e^{m+1}(m+1)^{-(m+1)/\mu}<\delta_0(B, U)\cdots\delta_m(B, U)\leqq(m+1)!\,\varepsilon^{m+1}M(B, U;\varepsilon)$$

on the basis of Lemma 4. If we put $\varepsilon=(m+1)^{-(1/\mu+1)}$, then the estimates $M(B, U;\varepsilon)\leqq\exp\{(m+1)^{\rho(1/\mu+1)}\}$ and $(m+1)!\leqq(m+1)^{m+1}$ together with multiplication by $(m+1)^{-(m+1)/\mu}$ and taking natural logarithmus lead to the contradiction $m+1<(m+1)^{\rho(1/\mu+1)}=m+1$.

Therefore, for each μ' with $\mu'>\mu$ we have

$$\sum_{n=n_0}^{\infty}\delta_n(B, U)^{\mu'}\leqq e\sum_{n=n_0}^{\infty}(n+1)^{-\mu'/\mu}<\infty.$$

Thus we obtain the inequality $\lambda\leqq\mu$, and so the second inequality in the lemma holds.

Remark. B. S. Mityagin [2] has proved Lemma 5 for a compact set B.

2. In this section we consider a locally convex topological vector space E over the field of real or complex numbers. Let U and V be two zero neighborhoods of E such that V is absorbed by U. Then, from Lemma 5 the following theorem holds.

Theorem 1. *Let λ be the exponent of convergence of the sequence $\{\delta_n(V, U)^{-1}\}$. Then $\rho(V, U) \leqq \lambda$. If $\rho(V, U) < 1$, then*

$$\lambda \leqq \rho(V, U)/\{1-\rho(V, U)\}.$$

A locally convex space E is called *$\mathfrak{s}$-nuclear* (cf. [4], p. 161) if for each zero neighborhood U of E, there exists a zero neighborhood V of E such that V is absorbed by U and the canonical mapping from $E(V)$ onto $E(U)$ is of type $\mathfrak{s}$. We have the following

Corollary. *A locally convex space E is $\mathfrak{s}$-nuclear if and only if each zero neighborhood U contains a zero neighborhood V such that $\rho(V, U)=0$.*

Proof. E is $\mathfrak{s}$-nuclear if and only if for each zero neighborhood U, there is a zero neighborhood V with $V \subset U$ such that the sequence $\{\delta_n(V, U)\}$ is rapidly decreasing by Lemma 1 of [3]. But $\{\delta_n(V, U)\}$ is rapidly decreasing if and only if $\lambda=0$, and $\lambda=0$ if and only if $\rho(V, U)=0$ by Theorem 5, where λ is the exponent of convergence of the sequence $\{\delta_n(V, U)^{-1}\}$.

Theorem 2. *Let U and V be two zero neighborhoods of a locally convex space E such that V is absorbed by U, and let $\tau(V, U)$ be the convergence type of the sequence $\{\delta_n(V, U)\}$ and*

$$\sigma(V, U)=\overline{\lim_{\varepsilon\to 0}}\,\{\log\log M(V, U\,;\,\varepsilon)/\log\log \varepsilon^{-1}\}.$$

Then we have

$$\sigma(V, U) \leqq \tau(V, U)+1.$$

Proof. Let $m(\varepsilon)=\sup\{n\,;\,\delta_n(V, U)>\varepsilon\}$ and $m(\varepsilon/4)=n-1$. Then $\delta_n(V, U) \leqq \varepsilon/4$. Therefore $M(V, U\,;\,\varepsilon) \leqq (4\delta_0(V, U)\varepsilon^{-1}+2)^n$ by Lemma 3. From this it follows that

$$\log\log M(V, U\,;\,\varepsilon) \leqq \log\{m(\varepsilon/4)+1\}+\log(\log 6\delta_0(V, U)+\log \varepsilon^{-1}).$$

But $\tau(V, U)=\overline{\lim}_{\varepsilon\to 0}\{\log m(\varepsilon/4)/\log\log \varepsilon^{-1}\}$ by Lemma 2. Thus the relation $\sigma(V, U) \leqq \tau(V, U)+1$ is proved.

Remark. For a vector space with a bornological structure, replacing two neighborhoods of the theorems above by two absolutely convex bounded sets, we can similarly show that the theorems above are valid for such a space.

References

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