# 78. *On K. Yosida's Class* (A) *of Meromorphic Functions*

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**1. Introduction.** The class $(A)$ in K. Yosida's sense [5] consists of all functions $f$ meromorphic in the plane $C: |z|<+\infty$ such that the family $\{f_\alpha\}, \alpha \in C$, of functions $f_\alpha(z)=f(z+\alpha), z\in C$, is normal in the sense of P. Montel in $C$. We set $k(f)=\sup_{z\in C} f^*(z)$ for $f\in(A)$, where $f^*(z)=|f'(z)|/(1+|f(z)|^2)$; we know that $k(f)<+\infty$ [5, Theorem 1]. Plainly, $k(f)>0$ if and only if $f$ is non-constant. Given a function $f$ meromorphic in $C$ and a point $z\in C$, let $u(z)=u(z,f)$ be the supremum of $r>0$ such that $f$ is univalent in the disk $D(z,r)=\{w\in C; |w-z|<r\}$; if such an $r$ does not exist, we set $u(z)=u(z,f)=0$. Then $u(z)=0$ if and only if $f^*(z)=0$. Except for the case that $f$ is linear, $u(z)<+\infty$ at each $z\in C$. Furthermore, a non-linear $f$ is univalent in $D(z,u(z))$ and the function $u$ is continuous in $C$ (Lemma). Here and elsewhere a meromorphic function $f$ is called non-linear if $f$ is non-constant and not linear. We begin with

**Theorem 1.** *Given a non-linear $f$ of class $(A)$, we have at each $z\in C$,*

$$f^*(z)\leqq(32/\pi^2)k(f)^2u(z,f). \tag{1}$$

Of course, the estimate (1) has the good meaning if $u(z,f)<\pi^2/\{32k(f)\}$. As an application of Theorem 1 we know that $u(z_n,f)\to 0$ implies $f^*(z_n)\to 0$ for each sequence of points $\{z_n\}\subset C$ converging to a point of $C$ or else to the point at infinity. However, the converse is not valid; the exponential function $E(z)=e^z$ belongs to $(A)$ with $u(z,E)=\pi$ at each $z\in C$ but $E^*(n)\to 0$ as $n\to+\infty$, $n$ being positive integers.

Our next result concerns the derived function.

**Theorem 2.** *Given a non-linear $f$ of class $(A)$, we have at each $z\in C$,*

$$f'^*(z)\leqq 2[\min\{k(f)^{-1}, u(z,f)\}]^{-1}+1, \tag{2}$$

*where* $f'^*(z)=|f''(z)|/(1+|f'(z)|^2)$.

The function $E\in(A)$ has the property that $E'\in(A)$, which suggests the following application of Theorem 2. We have $f'\in(A)$ if $f\in(A)$ and if $\inf_{|z|>R} u(z,f)>0$ for a certain constant $R>0$. Indeed, $f'^*$ is bounded in $|z|>R$ by (2), while $f'^*$ is bounded in $|z|\leqq 2R$ because $f'^*$ is continuous in $C$, whence $f'^*$ is bounded in $C$. Therefore $f'\in(A)$ by

[5, Theorem 1]. We remark that all rational functions (and their derivatives) belong to $(A)$.

We next consider the holomorphic case.

**Theorem 3.** *Given a non-linear entire function $f$, we have at each $z \in C$,*

$$f'^*(z) \leqq 2u(z, f)^{-1}. \tag{3}$$

We do not assume $f \in (A)$ in this case. Thus, if $f$ is non-linear and entire, and further $\inf_{|z|>R} u(z, f) > 0$ for some constant $R > 0$, we have $f' \in (A)$.

**2. Proofs.** First of all we need

**Lemma.** *For a non-linear meromorphic $f$ in $C$ we have for each pair $z, w \in C$,*

$$|u(z, f) - u(w, f)| \leqq |z - w|. \tag{4}$$

**Proof.** By the symmetry of $z$ and $w$ in (4) we have only to consider the case $u(z) < u(w)$. If $z \notin D(w, u(w))$, then $u(w) \leqq |z - w|$, whence follows (4). If $z \in D(w, u(w))$, then $f$ is univalent in $D(z, u(w) - |z - w|)$, from which follows $u(w) - |z - w| \leqq u(z)$, or $u(w) - u(z) \leqq |z - w|$, again (4).

**Proof of Theorem 1.** Since $u$ is continuous in $C$ by (4) and since $f^*$ is continuous in $C$, we have only to prove (1) for $z$ with $u(z) > 0$ and $f(z) \neq \infty$. Actually, the set of points $z \in C$ where $u(z) = 0$ or $f(z) = \infty$ is isolated. Set $b = \pi / \{4k(f)\}$ and consider the function of $w$:

$$g(w) = \frac{f(w+z) - f(z)}{1 + \overline{f(z)} f(w+z)} \tag{5}$$

in $D(0, b)$. Then $g(0) = 0, |g'(0)| = f^*(z)$ and $|g| < 1$ in $D(0, b)$. In effect, $g^*(\zeta) = f^*(\zeta + z) \leqq k(f), \zeta \in D(0, b)$, and

$$\begin{aligned} \operatorname{Arctan} |g(w)| &= \int_0^{|g(w)|} \frac{dt}{1+t^2} \leqq \int_{g(S_w)} \frac{|dw|}{1+|w|^2} \\ &= \int_{S_w} g^*(\zeta)\, |d\zeta| \leqq k(f)\, |w| < \pi/4, \end{aligned}$$

where $g(S_w)$ denotes the Riemannian image by $g$ of the line segment $S_w$ connecting 0 and $w \in D(0, b)$. Consider $h(\zeta) = g(b\zeta) / \{bg'(0)\}$ in $|\zeta| < 1$. Then $|h(\zeta)| < (b\,|g'(0)|)^{-1} = (bf^*(z))^{-1} = M$ in $|\zeta| < 1$ and $h(0) = 0, h'(0) = 1$. We may apply the result of J. Dieudonné (cf. [3, p. 259]) to $h$. Then $h$ is univalent in $|\zeta| < c = 1/(2M) < \{M + (M^2 - 1)^{1/2}\}^{-1}$, whence $g$ is univalent in $|w| < bc$, which implies $bc \leqq u(z)$. With a slight calculation we obtain (1).

**Proof of Theorem 2.** Since (2) is trivial if $u(z) = 0$ we may assume $u(z) > 0$. Moreover, by the continuity of $f'^*$ and $u$ we may again assume $f(z) \neq \infty$. Set $d = \min\{k(f)^{-1}, u(z)\}$. Then the function $g$ of (5) is univalent and holomorphic in $D(0, d)$. Actually, $g^*(\zeta) \leqq k(f)$ for each $\zeta \in D_1 = D(0, k(f)^{-1})$, whence

$$\operatorname{dis}(g(w), 0) \leqq \int_{S_w} g^*(\zeta)\,|d\zeta| \leqq k(f)\,|w| < 1$$

for each $w \in D_1$, dis $(\cdot, \cdot)$ being the chordal distance. Therefore $g$ has no pole in $D_1$. Consider $H(\zeta)=g(d\zeta)/\{dg'(0)\}$ in $|\zeta|<1$. Then $H(0)=0$, $H'(0)=1$ and $H$ is univalent in $|\zeta|<1$. Consequently, by the celebrated L. Bieberbach inequality $|a_2| \leqq 2$ for $H(\zeta)=\zeta+a_2\zeta^2+\cdots$, we have $|H''(0)| \leqq 4$ or $|g''(0)/g'(0)| \leqq 4/d$. After a short computation we obtain

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\overline{f(z)}f'(z)}{1+|f(z)|^2}\right| \leqq \frac{4}{d},$$

from which follows

$$f'^*(z)=\frac{|f''(z)|}{|f'(z)|}\,\frac{|f'(z)|}{1+|f'(z)|^2} \leqq \frac{4}{d}\,\frac{|f'(z)|}{1+|f'(z)|^2} + \frac{2|f(z)|}{1+|f(z)|^2}\,\frac{|f'(z)|^2}{1+|f'(z)|^2}$$
$$\leqq \frac{2}{d}+1,$$

because $t/(1+t^2) \leqq 1/2$ and $t^2/(1+t^2)<1$ for $t \geqq 0$. This completes the proof.

**Proof of Theorem 3.** We consider the function $G(w)=f(w+z)-f(z)$, $w \in D(0, u(z))$, for a fixed $z$ with $u(z)>0$. Then the function $F(\zeta)=G(u(z)\zeta)/\{u(z)f'(z)\}=\zeta+b_2\zeta^2+\cdots$ is univalent and holomorphic in $|\zeta|<1$. Hence, again,

$$|u(z)f''(z)/f'(z)|=|F''(0)|=2|b_2| \leqq 4,$$

whence $f'^*(z) \leqq 2u(z)^{-1}$.

**Remark.** K. Noshiro [1] obtained the notion of class $(A)$ in the disk $D=D(0,1)$ following the cited paper of Yosida; about twenty years after [1] O. Lehto and K. I. Virtanen discovered again the class $(A)$ in $D$ and called the members of $(A)$ normal meromorphic functions in $D$ (cf. [2, p. 86 ff.]). The results analogous to Theorems 1 and 2 for normal meromorphic functions in $D$ may easily be obtained and will be enunciated for the details elsewhere; the result analogous to (3) of Theorem 3 is seen in [4, (2)].

## References

[1] K. Noshiro: Contributions to the theory of meromorphic functions in the unit-circle. J. Fac. Sci. Hokkaido Univ., **7**, 149–159 (1938).

[2] ——: Cluster Sets. Berlin (1960).

[3] M. Tsuji: Potential Theory in Modern Function Theory. Tokyo (1959).

[4] S. Yamashita: The derivative of a holomorphic function in the disk (to be published in Michigan Math. J.).

[5] K. Yosida: On a class of meromorphic functions. Proc. Phys.-Math. Soc. Japan, **16**, 227–235 (1934).