74. A Generalization of Bieberbach's Example

By Masahide KATO
Department of Mathematics, Rikkyo University, Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., June 11, 1974)

1. Bieberbach constructed an example of a biholomorphic mapping of C^2 onto a proper open subset of C^2 ([1], see also [3]). His construction depends on the following fact. Let $g: z \to g(z)$ be a complex analytic automorphism of C^2 of which the origin 0 is a fixed point $g(0)=0$. The automorphism g induces a linear transformation of the tangent space $T_0(C^2)$ ($\simeq C^2$) of C^2 at 0. Assume that the eigenvalues α_1, α_2 of the linear transformation satisfy $1>|\alpha_1|\geq|\alpha_2|$. Then the set

$$U=\left\{z \in C^2 : \lim_{\nu\to+\infty} g^\nu(z)=0\right\}$$

is complex analytically isomorphic to C^2. The purpose of this paper is to generalize the above fact. Namely we shall prove

Theorem. *Let X be a complex space of dimension m. Assume that there exists a complex analytic automorphism g and a point $0 \in X$ such that $g(0)=0$ and $g^\nu(z)\to 0$ ($\nu\to+\infty$) for any point $z \in X$. Then X is complex analytically isomorphic to an affine variety. If, moreover, X is non-singular at 0, then $X \simeq C^m$.*

In [2], it is shown that the latter statement holds and that, if X is singular, X can be embedded into C^n as a closed subvariety which is invariant under a contracting complex analytic automorphism $\tilde{g}$ of C^n such that $\tilde{g}(0)=0$ and $\tilde{g}_{|X}=g$, where 0 denotes the origin of C^n. Let $(z_1, \cdots, z_n)$ be a standard system of coordinates of C^n. We may assume that $\tilde{g}$ has the following form;

$$
(1)\quad
\begin{aligned}
z_1' &= \alpha_1 z_1 \\
z_2' &= z_1 + \alpha_1 z_2 \\
&\vdots \\
z_{r_1}' &= z_{r_1-1} + \alpha_1 z_{r_1} \\
z_{r_1+1}' &= \alpha_2 z_{r_1+1} + P_{r_1+1}(z_1, \cdots, z_{r_1}) \\
&\vdots \\
z_{r_1+r_2}' &= z_{r_1+r_2-1} + \alpha_2 z_{r_1+r_2} + P_{r_1+r_2}(z_1, \cdots, z_{r_1}) \\
z_{r_1+r_2+1}' &= \alpha_3 z_{r_1+r_2+1} + P_{r_1+r_2+1}(z_1, \cdots, z_{r_1}, z_{r_1+1}, \cdots, z_{r_1+r_2}) \\
&\vdots \\
z_n' &= z_{n-1} + \alpha_\mu z_n + P_n(z_1, \cdots, z_{r_1+\cdots+r_{\mu-1}}),
\end{aligned}
$$

where $1>|\alpha_1|\geq|\alpha_2|\geq\cdots\geq|\alpha_\mu|>0$ and P_j ($r_1+\cdots+r_s<j\leq r_1+\cdots+r_{s+1}$) are finite sums of monomials $z_1^{m_1}\cdots z_{r_s}^{m_{r_s}}$ which satisfy $\alpha_{r_{s+1}}=\alpha_1^{m_1}\cdots\alpha_{r_s}^{m_{r_s}}$, $m_1+\cdots+m_{r_s}\geq 2$ and $m_l>0$ ([4], [5]).

2. Since the number of irreducible branches of X at 0 is finite, X has a finite number of irreducible components X_j $(j=1,2,\cdots)$. Hence there exists a positive integer l such that $\tilde{g}^l$ acts on each X_j as a contracting automorphism which has the similar form to (1). Therefore we may assume that X is an irreducible subvariety.

Lemma 1. *Let Z be a $\tilde{g}$-invariant subvariety in $\boldsymbol{C}^n$ such that $Z\supset X$ and $\dim Z>\dim X$. Then there exists a non-constant holomorphic function f on Z such that $\tilde{g}^*f=\alpha f$ $(0<|\alpha|<1)$ and $f_{|X}=0$.*

Proof. It is clear that both Z and X contain the origin $0\in\boldsymbol{C}^n$. Let D be a relatively compact neighborhood of 0 in Z such that $\tilde{g}(\bar{D})\subset D$, where $\bar{D}$ denotes the closure of D in Z. Let $\mathscr{B}$ be a vector space of holomorphic functions defined by

$$\mathscr{B}=\left\{\begin{matrix} f: & f \text{ is a bounded holomorphic function} \\ & \text{on } D \text{ such that } f_{|X\cap D}=0 \end{matrix}\right\}.$$

We define the norm $\|\ \|_D$ for $f\in\mathscr{B}$ by

$$\|f\|_D=\sup_{z\in D}|f(z)|.$$

Then $(\mathscr{B},\|\ \|_D)$ is clearly a Banach space. The linear mapping $\tilde{g}^*:\mathscr{B}\to\mathscr{B}$ defined by $(\tilde{g}^*f)(z)=f(\tilde{g}(z))$ is a compact operator by Vitali's theorem. It is easy to see that $\|\tilde{g}^*\|_D\le 1$ and $\|\tilde{g}^*f\|_D=\|f\|_D$ implies $f=0$. Now we shall show that there exists a non-zero element $f_0\in\mathscr{B}$ such that

$$\tilde{g}^*f_0=\alpha f_0 \qquad (0<|\alpha|<1).$$

Put

$$(2) \qquad R(\lambda)=(I-\lambda\tilde{g}^*)^{-1},$$

where I denotes the identity operator. Since $\tilde{g}^*$ is a compact operator, any spectrum except 0 is an eigenvalue ([7]). Hence if there is no such f_0, then $R(\lambda)$ is an entire function of λ on $\boldsymbol{C}$. This implies that the radius of the circle of convergence of the Taylor expansion of (2) is infinite, i.e., $\lim_{\nu\to+\infty}\sqrt[\nu]{\|\tilde{g}^{*\nu}\|_D}=0$. This is equivalent to saying that for any $\varepsilon>0$, there exists an integer ν_0 such that

$$(3) \qquad \|\tilde{g}^{*\nu}\|_D<\varepsilon^\nu$$

for $\nu>\nu_0$. Let $\mathbf{m}_{Z,0}$ denote the maximal ideal of $\mathcal{O}_{Z,0}$, ρ a positive integer such that there exists an element $h\in\mathscr{B}$ which is not contained in $\mathbf{m}_{Z,0}^{\rho+1}$. Fix a positive number ε such that $\varepsilon<|\alpha_i|^{\rho+1}$ for all i $(i=1,2,\cdots,\mu)$. Then

$$\|\varepsilon^{-\nu}\tilde{g}^{*\nu}h\|_D>\|h\|_D$$

for sufficiently large ν. But this contradicts (3). Hence there exists a non-zero element $f_0\in\mathscr{B}$ such that $\tilde{g}^*f_0=\alpha f_0$ $(0<|\alpha|<1)$. For every positive integer ν, we have

$$(4) \qquad f_0(z)=\alpha^{-\nu}f_0(\tilde{g}^\nu(z)) \qquad (z\in D).$$

Since $\alpha^{-\nu}\tilde{g}^{*\nu}f_0$ is defined on $\tilde{g}^{-\nu}(D)$ and $\bigcup_\nu\tilde{g}^{-\nu}(D)=Z$, it follows from (4) that f_0 can be continued analytically to a holomorphic function f on Z

such that $\tilde{g}^*f=\alpha f$. It is clear that $f_{|X}=0$. Q.E.D.

Denote by $\|z\|$ the norm of the point $z=(z_1, \cdots, z_n) \in \boldsymbol{C}^n$ defined by $\|z\|=|z_1|+\cdots+|z_n|$.

Lemma 2. *Let Z be a $\tilde{g}$-invariant subvariety in $\boldsymbol{C}^n$ and f a holomorphic function on Z which satisfies the equality*

$$\tilde{g}^*f=\alpha f \qquad (0<|\alpha|<1). \tag{5}$$

Then f satisfies the following inequality;

$$|f(z)|\leq M(1+\|z\|)^N, \tag{6}$$

where M and N are positive constants which are independent of $z \in Z$.

Proof. Let K be a closed small neighborhood of $0 \in \boldsymbol{C}^n$ defined by $\|z\|\leq\varepsilon$. First we estimate by $\|z\|$ the minimum integer ν such that $\tilde{g}^\nu(z) \in K$. By (1), the j-th coordinate $(r_1+\cdots+r_s<j\leq r_1+\cdots+r_{s+1})$ of the point $\tilde{g}^\nu(z)$ is given by

$$(\tilde{g}^\nu(z))_j=\alpha_{s+1}^\nu\{z_j+Q_j(\nu, z_1, \cdots, z_{j-1})\},$$

where Q_j is a polynomial of $\nu, z_1, \cdots, z_{j-1}$. Hence we get

$$\|\tilde{g}^\nu(z)\|\leq\sum_j |\alpha_{s+1}|^\nu \{|z_j|+|Q_j(\nu, z_1, \cdots, z_{j-1})|\}.$$

Then it is easy to see that, for some positive constants A, B and β ($|\alpha_i|<\beta<1$ for all i), the following inequality holds;

$$\|\tilde{g}^\nu(z)\|\leq A\beta^\nu(1+\|z\|)^B.$$

Let ν be the least integer such that $\nu>-(\log\beta)^{-1}\cdot\log(A(1+\|z\|)^B/\varepsilon)$. Then $\|\tilde{g}^\nu(z)\|\leq\varepsilon$, therefore $\tilde{g}^\nu(z) \in K$. Then, by (5),

$$\begin{aligned} |f(z)|&=|\alpha^{-\nu}f(\tilde{g}^\nu(z))| \\ &\leq|\alpha|^{-\nu}\|f\|_K \qquad (\|f\|_K=\sup_{z\in K}|f(z)|) \\ &\leq|\alpha|^{(\log\beta)^{-1}\cdot\log(A(1+\|z\|)^B/\varepsilon)-1}\cdot\|f\|_K \\ &=(A(1+\|z\|)^B/\varepsilon)^{(\log|\alpha|)/\log\beta}\cdot|\alpha^{-1}|\cdot\|f\|_K. \end{aligned}$$

Putting $N=B(\log|\alpha|)/\log\beta$ and $M=(A/\varepsilon)^{N/B}\cdot|\alpha^{-1}|\cdot\|f\|_K$, we get

$$|f(z)|\leq M(1+\|z\|)^N. \qquad \text{Q.E.D.}$$

W. Rudin [6] proved the following

Theorem 3. *An analytic subvariety V of pure dimension k in $\boldsymbol{C}^n$ is algebraic if and only if V lies in some algebraic region of type (k, n).*

By [6], a set Ω in $\boldsymbol{C}^n$ will be called an algebraic region of type (k, n) if there are vector subspaces E, F in $\boldsymbol{C}^n$ and a positive real numbers A, B such that the following conditions hold: $\dim E=k$, $\dim F=n-k$, $\boldsymbol{C}^n=E\oplus F$ (direct sum) and Ω consists precisely of the points $z \in \boldsymbol{C}^n$ satisfying the inequality

$$\|z''\|\leq A(1+\|z'\|)^B,$$

where $z=z'+z''$, $z' \in E$, $z'' \in F$.

Lemma 4. *Let Z be a $\tilde{g}$-invariant pure dimensional affine subvariety of $\boldsymbol{C}^n$, f a holomorphic function on Z such that $\tilde{g}^*f=\alpha f$ $(0<|\alpha|<1)$. Then the zero locus $Y=\{z \in Z: f(z)=0\}$ is a $\tilde{g}$-invariant affine subvariety of $\boldsymbol{C}^n$.*

Proof. It is sufficient to show that the graph

$$\Gamma_f=\{(z, w)\in Z\times \boldsymbol{C}: w=f(z)\}$$

is an affine subvariety of $\boldsymbol{C}^n\times \boldsymbol{C}=\boldsymbol{C}^{n+1}$. By Theorem 3, there exists an algebraic region of type (k, n) such that

$$(7) \qquad Z\subset\{(z', z'')\in \boldsymbol{C}^k\times \boldsymbol{C}^{n-k}: \|z''\|\leq A(1+\|z'\|)^B\},$$

where we can choose the subspace $\boldsymbol{C}^k$ so that there exists an algebraic branched covering $\theta: Z\to \boldsymbol{C}^k$. By (6), we have

$$\Gamma_f\subset\{(z, w)\in Z\times \boldsymbol{C}: |w|\leq M(1+\|z\|)^N\}\subset \boldsymbol{C}^{n+1}.$$

Hence, by (7), for points $(z, w)\in \Gamma_f$,

$$(8) \qquad \begin{aligned}|w|&\leq M(1+\|z'\|+\|z''\|)^N\leq M(1+\|z'\|+A(1+\|z'\|)^B)^N\\ &\leq M_1(1+\|z'\|)^{N_1},\end{aligned}$$

where M_1 and N_1 are some positive constants. Thus combining (8) with (7), we get

$$\begin{aligned}\|(z'', w)\|&\leq A(1+\|z'\|)^B+M_1(1+\|z'\|)^{N_1}\\ &\leq M_2(1+\|z'\|)^{N_2} \qquad \text{(for some } M_2 \text{ and } N_2).\end{aligned}$$

Hence the graph Γ_f is contained in an algebraic region of type $(k, n+1)$. Hence, by Theorem 3, Γ_f is an affine subvariety, since Γ_f is pure dimensional. Q.E.D.

Finally we prove the following

Lemma 5. *Let Z be a $\tilde{g}$-invariant pure dimensional affine subvariety of $\boldsymbol{C}^n$ such that $Z\supset X$ and $\dim Z>\dim X$. Then there exists a $\tilde{g}$-invariant pure dimensional affine subvariety Y of $\boldsymbol{C}^n$ such that $Z\supset Y\supset X$ and $\dim Z=\dim Y+1$. If $\dim Z=\dim X+1$, then X is an affine subvariety of $\boldsymbol{C}^n$.*

Proof. Let Z_0 be an irreducible component of Z such that $Z_0\supset X$. Put $W=\bigcup_{\nu\in Z}\tilde{g}^\nu(Z_0)$. Then W is a $\tilde{g}$-invariant pure dimensional affine subvariety of $\boldsymbol{C}^n$ which consists of the irreducible components of Z. Applying Lemmas 1 and 4 to W, we get a $\tilde{g}$-invariant affine subvariety Y defined by f which is a non-constant holomorphic function on W. Now Y contains no irreducible components of W. In fact, if Y contains an irreducible component of W, then f vanishes identically on W. Hence Y is pure dimensional and $\dim Y=\dim Z-1$. The latter statement is clear, since X is an irreducible component of the affine variety Y.

3. By Lemma 5, our main theorem can easily be proved by the induction on the codimension of X in Z. (Note that Z may be equal to $\boldsymbol{C}^n$.)

Acknowledgment. The author would like to express his hearty thanks to Prof. K. Saito who kindly read through the manuscript and pointed out that the original proof was incomplete.

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