102. *On Strongly Regular Rings. II*

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This is a natural sequel to [1] as well as to [2]. The notation and terminology employed there will be used here. In [2], R. Yue Chi Ming proved the following: *Let A be a ring with identity. If A is left non-singular and every finitely generated left ideal of A is the annihilator of a finitely generated left ideal then A is strongly regular, and conversely.* In this note, we shall prove an analogue without assuming the existence of identity. To this end, we shall introduce the following definition: A left ideal I of a ring A is called *quasi-finitely generated* if either I is finitely generated or $I=I'\oplus l(f)$ with a finitely generated left ideal I' and an idempotent f. Needless to say, if A is a ring with identity then every quasi-finitely generated left ideal is finitely generated.

Theorem. *The following conditions are equivalent*:

(i) *A ring A is strongly regular.*

(ii) *A is left non-singular and every quasi-finitely generated left ideal of A is the left annihilator of a quasi-finitely generated left ideal.*

In advance of the proof of our theorem, we state a couple of lemmas whose proofs are obvious by those of Lemmas 1 and 3 in [2].

Lemma 1. *The following conditions are equivalent*:

(i) *A ring A is left non-singular.*

(ii) *A is faithful as a right A-module and every left annihilator is closed in A.*

Lemma 2. *The following conditions are equivalent*:

(i) *A ring A is left non-singular and every closed left ideal of A is two-sided.*

(ii) *A contains no non-zero nilpotent element and $I+l(I)(=I\oplus l(I))$ is essential in A for every left ideal I of A.*

(iii) *A contains no non-zero nilpotent element and every closed left ideal of A is the left annihilator of a left ideal.*

Proof of Theorem. Assume first A is strongly regular. Obviously, A is then left non-singular. If I is an arbitrary quasi-finitely generated (left) ideal then one will easily see that $I=l(l(e))$ or $I=l(f-e)$ according as $I=Ae$ or $I=Ae\oplus l(f)$ with some central idempotents e and f. Conversely, assume (ii). Then, it is obvious that A is a left duo ring, and by Lemma 2 A contains no non-zero nilpotent element. If I is an arbitrary principal (left) ideal then by hypothesis $I=l(J)$ with some quasi-finitely generated left ideal J. Since J is a left annihilator again by hypothesis, we obtain $l(I)=l(l(J))=l(r(J))=J$. This proves that $I\oplus l(I)=I\oplus J$ is a quasi-finitely generated left ideal, and hence a left annihilator. Recalling here that A is left non-singular, $I\oplus l(I)$ is closed in A by Lemma 1. On the other hand, by Lemma 2 $I\oplus l(I)$ is essential in A. We have seen therefore $A=I\oplus l(I)$. Now, let a be an arbitrary element of A. Then, considering I as the left ideal generated by a^3, we have $a=u+v$, $u\in I$, $v\in l(I)$. Since $u^3+v^3=a^3\in I$, it follows then $v^3=0$, and hence $v=0$. This proves that $a\in I$ and A is strongly regular.

References

[1] K. Chiba and H. Tominaga: On strongly regular rings. Proc. Japan Acad., **49**, 435–437 (1973).
[2] R. Yue Chi Ming: On strongly regular rings (to appear).