98. *The Tensor Product of Weights*

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1. Introduction. The tensor product of normal semi-finite weights on von Neumann algebras was defined and used by several authors, e.g. F. Combes [3], A. Connes [6]. It was defined so that the resulting weight has favorable properties. Here in this note, we shall make a study on other possible definitions. We then establish a Radon-Nikodym type theorem for the tensor product of weights.

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2. The tensor product of normal semi-finite weights. Given a weight φ on a von Neumann algebra M, we denote by m_φ the *-subalgebra spanned by $n_\varphi^* n_\varphi$ where $n_\varphi=\{x \text{ in } M\,;\,\varphi(x^*x)<+\infty\}$. The linear extension on m_φ of $\varphi|_{(m_\varphi)_+}$ will be denoted by $\dot\varphi$. The following is the key lemma of our study.

Lemma 2.1. *Let a faithful normal semi-finite weight φ on M be given. Let τ be another normal semi-finite weight on M. If there exists a σ-weakly dense *-subalgebra B of m_φ, invariant under the modular automorphism group Σ of φ such that $\dot\varphi=\dot\tau$ on B, we have $\tau\leqq\varphi$, $\dot\tau|_{m_\varphi}=\dot\varphi$ and τ is faithful.*

Proof. The proof runs in the same way as in [5] Lemma 5.2. To get $\dot\tau|_{m_\varphi}=\dot\varphi$, we also make use of the expression $\tau(y^*x)=(\eta(x)\,|\,T\eta(y))$ for all x and y in n_φ as in [1] Lemma 2.3.

Let φ and ψ be faithful normal semi-finite weights on von Neumann algebras M and N. Let $\{\sigma_t\}$ and $\{\rho_t\}$ be the modular automorphism groups of φ and ψ, which will be denoted by Σ and Σ^ψ in what follows.

Proposition 2.2. *There exists a unique $\Sigma\otimes\Sigma^\psi$-invariant (i.e. $\sigma_t\otimes\rho_t$-invariant) normal semi-finite weight θ on $M\otimes N$ such that $\dot\theta\supset\dot\varphi\otimes_a\dot\psi$. Its modular automorphism group Σ^θ is the tensor product $\sigma_t\otimes\rho_t$. Furthermore if τ is a normal semi-finite weight on $M\otimes N$ such that $\tau\supset\dot\varphi\otimes_a\dot\psi$, we get $\dot\tau|_{m_\theta}=\dot\theta$ and τ is faithful.*

Proof. The existence is known ([6]. Definition 1.1.3). The uniqueness is due to [5] Proposition 5.9. Other properties are obtained from Lemma 2.1.

Theorem 2.3. *Let φ_1 and ψ_1 be normal semi-finite weights on M and N, p and q the support projections of φ_1 and ψ_1. There exists a unique normal semi-finite weight θ_1 on $M\otimes N$ such that $\dot\theta_1\supset\dot\varphi_1\otimes_a\dot\psi_1$ and θ_1 is $\Sigma^{\varphi_1}\otimes\Sigma^{\psi_1}$-invariant on the von Neumann algebra $p\otimes q(M\otimes N)p\otimes q$.*

Its modular automorphism group Σ^{θ_1} *on* $p\otimes q(M\otimes N)p\otimes q$ *is* $\sigma_t^{\varphi_1}\otimes\sigma_t^{\psi_1}$. *Furthermore for every normal weight* τ_1 *with* $\dot{\tau}_1\supset\dot{\varphi}_1\otimes_a\dot{\psi}_1$, *we get* $\dot{\tau}_1|_{m_{\theta_1}}=\dot{\theta}_1$ *and its support projection is* $p\otimes q$.

Definition 2.4. The weight θ_1 obtained in Theorem 2.3 is called the tensor product of φ_1 and ψ_1, which will be denoted by $\varphi_1\otimes\psi_1$.

F. Combes has given the notion of strict semi-finiteness of a normal weight and has established that, given normal strictly semi-finite weights φ,ψ resp. on M,N resp., a unique normal strictly semi-finite weight can be constructed on $M\otimes N$ as the tensor product of φ,ψ. This tensor product is nothing other than that which is obtained by our definition. The proof leans on the fact that our tensor product has some maximal property as shown in Theorem 2.3. Conversely, if φ and ψ are non-zero faithful normal semi-finite weights on M,N, and if $\varphi\otimes\psi$ is strictly semi-finite, then both φ and ψ are strictly semi-finite.

3. The Radon-Nikodym theorem on the tensor product.

Theorem 3.1. *Let* φ *and* ψ *be faithful normal semi-finite weights on M and N. Let* φ_1 *and* ψ_1 *be other normal semi-finite weights on M and N such that* $\varphi_1=\varphi(h\cdot)$ *and* $\psi_1=\psi(k\cdot)$ *where h and k are positive self-adjoint operators affiliated with the subalgebras* M^{Σ} *and* N^{Σ^ψ} *of fixed points of M and N for* Σ *and* Σ^ψ *respectively. Then we get* $\varphi_1\otimes\psi_1(\cdot)=\varphi\otimes\psi(h\otimes k\cdot)$.

For the proof, we need the following definition and a lemma.

Definition 3.2. If h and k are positive self-adjoint operators on a Hilbert space H and $\varepsilon>0$, we put $h_\varepsilon=h(1+\varepsilon h)^{-1}$. We write $h\leqq k$ if $h_\varepsilon\leqq k_\varepsilon$ for some (and hence for any) $\varepsilon>0$. We say that a net $\{h_i\}$ of positive self-adjoint operators increases to a self-adjoint h and write $h_i\nearrow h$ if $(h_i)_\varepsilon\nearrow h_\varepsilon$. (See [5].)

Lemma 3.3. *Let h (resp. k) be a positive self-adjoint operator on a Hilbert space* H_1 *(resp.* H_2*). We get* $h_\delta\otimes k_\varepsilon\nearrow h\otimes k$ *when* $\delta\searrow 0,\varepsilon\searrow 0$.

Proof of Theorem 3.1. For each x in $(m_{\varphi_1})_+$ and y in $(m_{\psi_1})_+$, we have

$$\varphi_1(x)=\lim_\varepsilon\varphi(h_\varepsilon\cdot x),\qquad \psi_1(y)=\lim_\delta\psi(k_\delta\cdot y).$$

By [5] Proposition 4.2 we have

$$h_\varepsilon^{1/2}xh_\varepsilon^{1/2}\text{ is in }m_\varphi,\ k_\delta^{1/2}yk_\delta^{1/2}\text{ is in }m_\psi$$

for all $\varepsilon>0,\delta>0$. Therefore

$$\varphi_1\otimes\psi_1(x\otimes y)=\lim_{(\varepsilon,\delta)}\varphi\otimes\psi(h_\varepsilon\otimes k_\delta\cdot x\otimes y).$$

By Lemma 3.3 and [5] Proposition 4.2 we get

$$\varphi_1\otimes\psi_1(x\otimes y)=\varphi\otimes\psi(h\otimes k\cdot x\otimes y)$$

for all x in $(m_{\varphi_1})_+$, y in $(m_{\varphi_1})_+$. By [5] Theorem 4.6 $\sigma_t^{\varphi_1\otimes\psi_1}(\cdot)=(h^{it}\otimes k^{it})\sigma_t\otimes\sigma_t^\psi(\cdot)(h^{-it}\otimes k^{-it})$. Therefore $\varphi\otimes\psi(h\otimes k\cdot)$ is $\sigma_t^{\varphi_1\otimes\psi_1}$-invariant on $[h]\otimes[k](M\otimes N)[h]\otimes[k]$ where $[h]$ and $[k]$ are the range projections of h and k respectively.

Thus we get $\varphi_1 \otimes \psi_1 = \varphi \otimes \psi(h \otimes k \cdot)$ by virture of Theorem 2.4.

References

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