94. *On Strongly Pseudo-Convex Manifolds*

By Noboru TANAKA
Department of Mathematics, Kyoto University

(Comm. by Kôsaku YOSIDA, M. J. A., Sept. 12, 1974)

By a strongly pseudo-convex (s.p.c) manifold we mean the abstract model (cf. Kohn [2]) of a s.p.c. real hypersurface of a complex manifold. The main aim of this note is to announce some theorems on compact s.p.c. manifolds M, especially on the cohomology groups $H^{p,q}(M)$ due to Kohn-Rossi [3] and the holomorphic de Rham cohomology groups $H_0^k(M)$ (see Theorems 1, 2). We also apply Theorem 2 to the study of isolated singular points of complex hypersurfaces (see Theorem 4).

Throughout this note we always assume the differentiability of class C^∞. Given a fibre bundle E over a manifold M, $\Gamma(E)$ denotes the set of differentiable cross sections of E.

1. S.p.c. manifolds. Let M' be an n-dimensional complex manifold and M a real hypersurface of M'. Let T' (resp. T) be the complexified tangent bundle of M' (resp. of M). Denote by S' the subbundle of T' consisting of all tangent vectors of type $(1,0)$ to M' and, for each $x\in M$, put $S_x=T_x\cap S'_x$. Then we have $\dim_C S_x=n-1$ and hence the union $S=\bigcup_x S_x$ forms a subbundle of T. It is easy to see that S satisfies

1) $S\cap\bar{S}=0$,

2) $[\Gamma(S),\Gamma(S)]\subset\Gamma(S)$.

By 1), the sum $P=S+\bar{S}$ is a subbundle of T. Consider the factor bundle $Q=T/P$ and denote by ϖ the projection of T onto Q. For each $x\in M$, define an Q_x-valued quadratic form H_x on S_x, the Levi form at x, by $H_x(X_x)=\varpi([X,\bar{X}]_x)$ for all $X\in\Gamma(S)$. Then M is, by definition, s.p.c. if S satisfies

3) the Levi form H_x is definite at each $x\in M$.

Let M be a (real) manifold of dimension $2n-1$. Suppose that there is given an $(n-1)$-dimensional subbundle S of the complexified tangent bundle T of M. Then S is called a s.p.c. structure if it satisfies conditions 1), 2) and 3) stated above, and the manifold M together with the structure is called a s.p.c. manifold.

2. The cohomology groups $H^{p,q}(M)$, $H_0^k(M)$ and $H_*^{p,q}(M)$. Let M be a s.p.c. manifold of dimension $2n-1$ and S its s.p.c. structure. Let $\{\mathcal{A}^k, d\}$ be the de Rham complex of M with complex coefficients.

For any integers p and k, denote by $F^p(\mathcal{A}^k)$ the subspace of $\mathcal{A}^k$ consisting of all $\varphi \in \mathcal{A}^k$ which satisfy

$$\varphi(X_1, \cdots, X_{p-1}, \bar{Y}_1, \cdots, \bar{Y}_{k-p+1})=0$$

for all $X_1, \cdots, X_{p-1} \in T_x, Y_1, \cdots, Y_{k-p+1} \in S_x$ and $x \in M$. Then we easily find that the system $\{F^p(\mathcal{A}^k)\}$ gives a filtration of the de Rham complex. Note that the filtration is canonically bounded, i.e., $F^0(\mathcal{A}^k)=\mathcal{A}^k$ and $F^{p+1}(\mathcal{A}^p)=0$. Let $\{E_r^{p,q}\}$ denote the spectral sequence associated with the filtration.

The groups $H^{p,q}(M)$. We denote by $H^{p,q}(M)$ the groups $E_1^{p,q}$ which are the cohomology groups associated with the complexes $\{C^{p,q}, \bar\partial\}$, where $C^{p,q}=F^p(\mathcal{A}^{p+q})/F^{p+1}(\mathcal{A}^{p+q})$ and the operator $\bar\partial: C^{p,q} \to C^{p,q+1}$ is naturally induced from the operator $d: F^p(\mathcal{A}^{p+q}) \to F^p(\mathcal{A}^{p+q+1})$. It can be shown that the space $C^{p,q}$ may be described as $\Gamma(\wedge^p \hat{S}^* \otimes \wedge^q \bar{S}^*)$, where $\hat{S}=T/\bar{S}$. (Suppose that the s.p.c. manifold M is realized as a s.p.c. hypersurface of a complex manifold M'. Then it is easy to see that the complexes $\{C^{p,q}, \bar\partial\}$ coincide with the complexes $\{\mathcal{B}^{p,q}, \bar\partial_b\}$ introduced by Kohn-Rossi [3]. Note that they erroneously described the space $C^{p,q}$ as $\Gamma(\wedge^p S^* \otimes \wedge^q \bar{S}^*)$.) In the same manner as Kohn [2], we can develop the harmonic theory for the complexes $\{C^{p,q}, \bar\partial\}$. In particular we have $\dim H^{p,q}(M) < \infty \ (q \neq 0, n-1)$, provided M is compact.

The groups $H_0^k(M)$. The group $E_1^{k,0}=\{\varphi \in C^{k,0} \mid \bar\partial\varphi=0\}$ is called the space of holomorphic k-forms. We denote by $H_0^k(M)$ the groups $E_2^{k,0}$, which are the cohomology groups associated with the complex $\{E_1^{k,0}, d\}$, the holomorphic de Rham complex.

The groups $H_*^{p,q}(M)$. If we put $\mathcal{A}^{p,q}=F^p(\mathcal{A}^{p+q})$, we have $d\mathcal{A}^{p,q} \subset \mathcal{A}^{p,q+1}$. Thus the systems $\{\mathcal{A}^{p,q}, d\}$ form complexes. We denote by $H_*^{p,q}(M)$ the cohomology groups associated with these complexes.

The short exact sequences

$$0 \to \mathcal{A}^{k,q} \to \mathcal{A}^{k-1,q+1} \to C^{k-1,q+1} \to 0$$

induce the exact sequences of cohomology groups

$$(*) \qquad 0 \to H_0^k(M) \to H_*^{k-1,1}(M) \to H^{k-1,1}(M) \to H_*^{k,1}(M) \to .$$

Since $H_0^{n+1}(M)=H_*^{n+1,1}(M)=0$, it follows that $H_*^{n,1}(M) \cong H^{n,1}(M)$. Consequently we get the exact sequence

$$(*') \qquad 0 \to H_0^n(M) \to H_*^{n-1,1}(M) \to H^{n-1,1}(M) \to H^{n,1}(M).$$

3. Finiteness for the groups $H_*^{k-1,1}(M)$ and $H_0^k(M)$. Let M be a compact s.p.c. manifold. We assume that $\dim M=2n-1 \geq 5$. Let k be any integer. By its definition $H_*^{k-1,1}(M)$ was the cohomology group associated with the complex

$$\mathcal{A}^{k-1,0} \xrightarrow{d} \mathcal{A}^{k-1,1} \xrightarrow{d} \mathcal{A}^{k-1,2}.$$

We take a Riemannian metric g on M, which gives rise to inner products (,) in the spaces $\mathcal{A}^{k-1,i}$ $(i=0,1,2)$. Let δ denote the adjoint operators of d with respect to these inner products. We also define Sobolev norms $\| \ \|_s$, s being any real number, in the spaces $\mathcal{A}^{k-1,i}$.

Theorem 1. *The Laplacian $\Delta=\delta d+d\delta: \mathcal{A}^{k-1,1}\to\mathcal{A}^{k-1,1}$ is subelliptic, that is, there is a positive number σ such that*

$$\|\varphi\|_\sigma^2 \leqq C((\Delta\varphi,\varphi)+\|\varphi\|_0^2) \qquad (\varphi\in\mathcal{A}^{k-1,1}),$$

where C is a positive constant independent of φ.

Corollary. $\dim H_0^k(M)\leqq \dim H_*^{k-1,1}(M)<\infty$ (*by exact sequence* $(*)$).

Now let M be a compact manifold of dimension $2n-1\geqq 5$. Suppose that there is given a differentiable family $\{S(t)\}_{t\in T}$ of s.p.c. structures on M, the parameter space T being a domain in the space R^l of l real variables. Let $M(t)$ denote the s.p.c. manifold M with the structure $S(t)$.

Theorem 2. *The integer valued function $\rho^k(t)=\dim H_*^{k-1,1}(M(t))$ $(t\in T)$ is upper semi-continuous.*

4. Isolated singular points of complex hypersurfaces. We first state the following

Proposition 3. *Let M' be a complex manifold of dimension $n\geqq 3$, and V a relatively compact subdomain of it. Assume that V is a Stein manifold and that the boundary $M=\partial V$ of V is a smooth, compact, connected, s.p.c. hypersurface of M'. Then we have*

(1) $$H^{p,q}(M)=0 \qquad (q\neq 0, n-1),$$

(2) $$H_0^k(M)\cong H^k(V),$$

where $H^k(V)$ denotes the k-th de Rham cohomology group of V.

(1) is due to Kohn-Rossi [3]. The proof of (2) above all uses the fact that $H^{p,q}(\bar V)=H^{p,q}(V)=0$ $(q\neq 0)$ (Kohn [1]), where $\bar V=V\cup M$.

Let f be a polynomial function on the space C^{n+1} of $n+1$ complex variables, where $n\geqq 3$. We assume that f vanishes at the origin and that the origin is an isolated critical point of f. Let V be the complex hypersurface defined by $f=0$ and S_ε^m $(m=2n+1)$ the ε-sphere in C^{n+1} centred at the origin. We put $M_\varepsilon=V\cap S_\varepsilon^m$. Then, for ε sufficiently small, M_ε is a compact, connected, s.p.c. hypersurface of V (cf. Milnor [4]).

Theorem 4. *Let μ be the multiplicity of the isolated singular point, the origin, of the complex hypersurface V (Milnor [4]). Then we have, for ε sufficiently small,*

$$\mu\leqq \dim H_*^{n-1,1}(M_\varepsilon)\leqq \dim H_0^n(M_\varepsilon)+\dim H^{n-1,1}(M_\varepsilon).$$

Proof. For $c\in C$, let $V(c)$ denote the complex hypersurface defined by $f=c$. We put $V_\varepsilon(c)=V(c)\cap B_\varepsilon^m$ and $M_\varepsilon(c)=V(c)\cap S_\varepsilon^m$, where B_ε^m is the open ε-ball in C^{n+1} centred at the origin. Note that $M_\varepsilon=M_\varepsilon(0)$ and that $M_\varepsilon(c)$ is the boundary of the domain $V_\varepsilon(c)$ in $V(c)$. From now on, ε and c will be such that $\varepsilon<\delta$ and $|c|<\delta$, δ being sufficiently small. Then we see that $M_\varepsilon(c)$ is a compact, connected, s.p.c. hypersurface of

$V(c)$ and that $V_\varepsilon(c)$ $(c\neq 0)$ is a Stein manifold. Consequently by Proposition 3, (1), we have $H^{n-1,1}(M_\varepsilon(c))=0$ $(c\neq 0)$ and hence by Proposition 3, (2) and exact sequence $(*')$, $H^n_\varepsilon(V_\varepsilon(c))\cong H^n_0(M_\varepsilon(c))\cong H^{n-1,1}_*(M_\varepsilon(c))$ $(c\neq 0)$. Therefore it follows from Milnor [4] that $\dim H^{n-1,1}_*(M_\varepsilon(c)) =\dim H^n(M_\varepsilon(c))=\mu$ $(c\neq 0)$. Furthermore we see that, ε being fixed, the family $\{M_\varepsilon(c)\}_{|c|<\delta}$ is a differentiable family (or a deformation) of s.p.c. manifolds. Therefore by Theorem 2, we have $\dim H^{n-1,1}_*(M_\varepsilon) \geqq \dim H^{n-1,1}_*(M_\varepsilon(c))=\mu$ $(c\neq 0)$, proving Theorem 4.

For example, consider the case where $f(z_1,\cdots,z_{n+1})=z_1^2+\cdots+z_{n+1}^2$. Then we have $\mu=1$. Furthermore we can show that $H^{p,q}(M_\varepsilon) =0$ $(p+q\neq n-1,n;\ q\neq 0,n-1)$ and $H^k_0(M_\varepsilon)=0$ for all k. Hence $H^{n-1,1}_*(M_\varepsilon)\cong H^{n-1,1}(M_\varepsilon)$ and $\dim H^{n-1,1}(M_\varepsilon)\geqq 1$.

References

[1] J. J. Kohn: Harmonic integrals on strongly pseudo-convex manifolds. I. Ann. of Math., **78**, 112–148 (1963); II: ibid., **79**, 450–472 (1964).

[2] ——: Boundaries of Complex Manifolds. Proc. Conference on Complex Manifolds (Minneapolis), Springer-Verlag, New York (1965).

[3] J. J. Kohn and H. Rossi: On the extension of holomorphic functions from the boundary of a complex manifold. Ann. of Math., **81**, 451–472 (1965).

[4] J. Milnor: Singular Points of Complex Hypersurfaces. Ann. Math. Studies No. 61, Princeton Univ. Press, Princeton, New Jersey (1968).