92. On the Structure of Certain Types of Polarized Varieties. II

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This is a continuation of our previous notes [1], [2]. We employ the same notation and the same terminology as in them. We shall outline our main results. Details will be published elsewhere.

1. Polarized varieties with $\Delta=0$. Given a pair (x, y) of points on a projective space $\boldsymbol{P}$, we denote by $l_{x,y}$ the line which passes through the points x and y. Given a pair (X, Y) of subsets of $\boldsymbol{P}$, we denote by X^*Y the subset $(\bigcup_{(x,y)\in X\times Y, x\neq y} l_{x,y}) \cup X \cup Y$ of $\boldsymbol{P}$.

Theorem 1. i) *Let (V, F) be a polarized variety with $\Delta(V, F)=0$. Then V is normal and F is very ample.*

ii) *Let $\rho: V \to \boldsymbol{P}^N$ be the embedding associated with F, and let S be the set of singular points of V. Then S is a linear subspace of $\boldsymbol{P}^N$.*

iii) *Let L be a linear subspace of $\boldsymbol{P}^N$ such that* $\dim L+\dim S=N-1$ *and $L\cap S=\emptyset$. Put $V_L=V\cap L$. Then V_L is non-singular, $\Delta(V_L, F)=0$ and $V=V_L{}^*S$.*

Remark. By this theorem the classification of polarized varieties with $\Delta=0$ is reduced to that of non-singular ones. Recall that an enumeration of such polarized manifolds has already been given in [1].

2. Families of polarized varieties with $\Delta=0$. Theorem 2. *Let $\pi: \mathcal{V} \to T$ be a proper, flat morphism from a variety V to another variety T, which may not be compact. Suppose that for every $t\in T$ the fiber $V_t=\pi^{-1}(t)$ is irreducible and reduced. Let F be a line bundle on $\mathcal{V}$ which is relatively ample to π. Suppose that $\Delta(V_0, F_0)=\Delta(V_0, F_{V_0})=0$ for some $0\in T$. Then $\Delta(V_t, F_t)=0$ for any $t\in T$.*

Corollary 2.1. *Suppose in addition that $d(V_0, F_0)=1$. Then $\mathcal{V}$ is a $\boldsymbol{P}^n$-bundle over T.*

Corollary 2.2. *Suppose in addition that $d(V_0, F_0)=2$. Then there exists an embedding $\mathcal{V}\to\mathcal{P}$ where $\mathcal{P}$ is a $\boldsymbol{P}^{n+1}$-bundle over T. Moreover $\mathcal{V}$ is a divisor on $\mathcal{P}$ and V_t is a quadric in $P_t\cong\boldsymbol{P}^{n+1}$ which is the fiber of $\mathcal{P}\to T$ over $t\in T$.*

Corollary 2.3. *Suppose in addition that $d(V_0, F_0)\geqq 3$, that V_0 is non-singular and that the canonical bundle of V_0 is a restriction of a line bundle on $\mathcal{V}$. Then every fiber V_t is non-singular. Moreover, except the case in which $\mathcal{V}$ is a $\boldsymbol{P}^2$-bundle over T, there exists a $\boldsymbol{P}^1$-*

bundle $\mathcal{W}$ over T such that $\mathcal{V}$ is a $\boldsymbol{P}^{n-1}$-bundle over $\mathcal{W}$.

3. Certain polarized manifolds with $\Delta=1$, $d=1$. **Lemma.** *Let (V,F) be a polarized variety with $\dim V=1$, $d(V,F)=2$, $\Delta(V,F)=1$. Then $Bs|F|=\emptyset$ and the morphism associated with $|F|$ makes V a two-sheeted branched covering of $\boldsymbol{P}^1$.*

Let F be a line bundle on a manifold M and let B be a non-singular member of $|kF|$ where k is an integer, $k\geqq 2$. Then there exists a submanifold N of F such that the bundle mapping $F\to M$ makes N a k-sheeted branched covering of M with branch locus B. Such a manifold N is determined uniquely by the quadruple (M,k,B,F) up to isomorphism. (See [4].) We denote N by $R_{k,B,F}(M)$. We write $R_B(M)$ for $R_{2,B,F}(M)$ if there is no danger of confusion.

Theorem 3. i) *Let (M,F) be a polarized manifold with $\Delta(M,F)=1$, $d(M,F)=1$, $\dim M=n$ and $g(M,F)\leqq 2$. Then there exists a vector bundle E on $\boldsymbol{P}^{n-1}$ of rank 2 and a non-singular divisor B on $P=\boldsymbol{P}(E^*)$ such that $Q_p(M)=R_B(P)$ where p is the base point of $|F|$ (see [1], Proposition F).*

ii) *Let H and I denote the hyperplane bundle and trivial bundle of $\boldsymbol{P}^{n-1}$ respectively and put $L=-L(E^*)$.*

a) *Suppose that $g(M,F)=1$, then $E=I\oplus 2H$, $B=B_1+B_2$, $B_1\in|L-2H|$, $B_2\in|3L|$.*

b) *When $g(M,F)=2$, one of the following cases occurs:*

b-0) *$E=I\oplus I$, B is connected and $B\in|6L+2H|$,*

b-1) *$E=I\oplus H$, B is connected and $B\in|6L|$,*

b-2) *$E=I\oplus 2H$, $B=B_1+B_2$, $B_1\in|L-2H|$, $B_2\in|5L|$.*

When $\dim M\geqq 4$, b-2) is the case.

Corollary. *$H^1(M,\mathcal{O}_M)=0$ if $\dim M\geqq 2$.*

Remark. A polarized surface of the above type b) is a rational surface or a blowing-up of a $K3$-surface or a surface of general type according as it is of type b-0), b-1) or b-2).

References

[1] T. Fujita: On the structure of certain types of polarized varieties. Proc. Japan Acad., **49**, 800–802 (1973).

[2] ——: On some new invariants of polarized manifolds. Proc. Japan Acad., **50**, 173–174 (1974).

[3] ——: On polarized manifolds with Δ-genera two (to appear).

[4] J. Wavrik: Deformations of branched coverings of complex manifolds. Amer. J. Math., **90**, 926–960 (1968).