91. *Extension Theorems for Kähler Metrics*

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Let X be a complex manifold and let $\hat{X}$ denote a monoidal transform of X of which the center is a point. The folowing proposition is well-known:

If X admits a Kähler metric, then $\hat{X}$ also admits a Kähler metric.

In this note we shall prove an extension theorem for Kähler metrics of which the converse of the above proposition is a corollary. Moreover we shall show a similar extension theorem for a certain type of branched coverings.

1. Formulation of the results. In this section we denote by X a complex manifold and let $D_r^n = \{z \in \boldsymbol{C}^n \mid z < r\}$.

Proposition A. *Assume that $D_r^n - 0$ has a Kähler form ω. Then D_r^n admits a Kähler form $\tilde{\omega}$ such that $\tilde{\omega} = \omega$ on $D_r^n - D_{2r/3}^n$.*

Corollary. *Let P denote a point on X. If $X - P$ is a Kähler manifold, then X is also a Kähler manifold.*

Let $\tilde{D} = D_r^1 \to D = D_{r^m}^1$ be the m-fold branched covering defined by the mapping $z \to z^m$, and let Γ denote the covering transformation group of $\tilde{D}$ with respect to D. Moreover let $p: X \to D$ be a surjective proper smooth holomorphic mapping, $\tilde{X} = X \times_D \tilde{D}$, and denote by π the induced covering map: $\tilde{X} \to X$. The group Γ acts on $\tilde{X}$ in an obvious manner.

Proposition B. *If $\tilde{X}$ has a Γ-invariant Kähler metric $\tilde{\omega}$, then X admits a Kähler metric ω such that $\tilde{\omega} = \pi^* \omega$ on $\pi^{-1} p^{-1} (D - D_{(2r/3)^m}^1)$.*

Corollary. *Let Δ and $\tilde{\Delta}$ be compact Riemann surfaces, X a compact complex manifold of dimension n, and let $p: X \to \Delta$ be a fibre manifold. Moreover let $\pi: \tilde{\Delta} \to \Delta$ be a finite Galois covering. Let $\tilde{X}$ denote the normalization of the fibre product $X \times_\Delta \tilde{\Delta}$. Assume that the induced covering $\tilde{X} \to X$ has its branch locus on regular fibres of $p: X \to \Delta$. Then X is a Kähler manifold if and only if $\tilde{X}$ is a Kähler manifold.*

2. Proof of Proposition A. By $\mathcal{D}$ and $\mathcal{F}$ we denote, respectively, the sheaves of differentiable functions and differentiable d-closed $(1,1)$-forms. We have a natural exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O} + \overline{\mathcal{O}} \longrightarrow \mathcal{D} \xrightarrow{\sqrt{-1}\partial\bar{\partial}} \mathcal{F} \longrightarrow 0.$$

Lemma 1. *Let ω be a d-closed $(1,1)$-form on $W = D_r^n - 0$, $n > 2$.*

Then there exists a differentiable function u on W such that

$$\omega=\sqrt{-1}\partial\bar{\partial}u.$$

Proof. In view of the exact sequence

$$H^0(W,\mathcal{D})\to H^0(W,\mathcal{F})\to H^1(W,\mathcal{O}+\bar{\mathcal{O}}),$$

the assertion is reduced to the equality $H^1(W,\mathcal{O}+\bar{\mathcal{O}})=0$, which follows immediately from Scheja's theorem (cf. [5] Satz 1).

Lemma 2 (Shiffman). *Let*

$$\omega=\sqrt{-1}\partial\bar{\partial}w_{\alpha\beta}(z)dz^\alpha\wedge d\bar{z}^\beta$$

be a real d-closed (1, 1)*-form on the domain* $W=D_1^2-0$. *If* $w_{11}\geqq 0$ *and* $w_{22}\geqq 0$ *on W, then there exists a real differentiable function u(z) on W such that*

$$\omega=\sqrt{-1}\partial\bar{\partial}u.$$

For a proof, see [6] p. 333.

From Lemma 1 and Lemma 2, we obtain the following

Lemma 3. *Let* ω *be a Kähler form on* $W=D_r^n-0$ $(n\geqq 2)$. *Then there exists a strictly plurisubharmonic function u on W such that*

$$\omega=\sqrt{-1}\partial\bar{\partial}u.$$

Lemma 4 (Grauert-Remmert). *Let V be a closed analytic subvariety of a domain U. A plurisubharmonic function u on* $U-V$ *can be extended to a plurisubharmonic function defined on U if either the codimension of V is greater than* 1 *or u is bounded from above.*

For a proof, see [1] Satz 3 and 4.

Proof of Proposition A. If $n=1$, any hermitian metric is Kähler and the assertion is trivial. Assume that $n>1$. Then by means of Lemmas 3 and 4, we find a plurisubharmonic function $u(z)$ on D_r^n such that $\omega=\sqrt{-1}\partial\bar{\partial}u$ on D_r^n-0. Let $\lambda(z)$ be a non-negative differentiable function defined on C^n such that

(i) $\lambda(z)$ depends only on $|z|$,

(ii) $\operatorname{supp}\lambda\subset D_{2r/3}^n$,

(iii) $\int_{C^n}\lambda(z)dv(z)=1$, where $dv(z)$ denotes the volume form,

(iv) λ is a positive constant on $D_{r/3}^n$.

Then, for a small positive number ε,

$$\tilde{u}(z)=\int_{C^n}u(z-\varepsilon\lambda(z)\zeta)\lambda(\zeta)dv(\zeta)$$

is a differentiable function on D_r^n. We infer readily that $\tilde{u}$ converges uniformly to u on $D_r^n-D_{r/3}^n$ for $\varepsilon\downarrow 0$ in the C^2-sense, and that $\tilde{u}=u$ on $D_r^n-D_{2r/3}^n$. Therefore, for sufficiently small ε, $\tilde{u}$ is strictly plurisubharmonic on $D_r-D_{r/3}$. On the other hand, since $\lambda(z)$ is constant for $z\in D_{r/3}$, it follows from a well-known theorem that $\tilde{u}$ is strictly plurisubharmonic on $D_{r/3}$ (Hörmander [2] p. 45). Hence $\tilde{\omega}=\sqrt{-1}\partial\bar{\partial}\tilde{u}$ is a Kähler metric which satisfies the conditions.

3. Proof of Proposition B. Shrinking D if necessary, we can choose open coverings:

$$X=\cup U_i, \qquad U_i \cong D\times D_\delta^{n-1},$$
$$\tilde{X}=\cup \tilde{U}_i, \qquad \tilde{U}_i \cong \tilde{D}\times D_\delta^{n-1}.$$

Then there exists a strictly plurisubharmonic function $\tilde{u}_i$ on $\tilde{U}_i$, such that $\omega=\sqrt{-1}\partial\bar{\partial}\tilde{u}_i$. Replacing $\tilde{u}_i$ by $\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma^*\tilde{u}_i$, where $|\Gamma|$ denotes the order of Γ, we may assume that $\tilde{u}_i$ is Γ-invariant. Therefore $\tilde{u}_i$ can be regarded as a continuous plurisubharmonic function on U_i. $w_{ij}=\tilde{u}_i-\tilde{u}_j$ is continuous on $U_i\cap U_j$ and is harmonic on $U_i\cap U_j - U_i\cap U_j\cap p^{-1}(0)$. Hence w_{ij} is a harmonic function on $U_i\cap U_j$. Let $x, (\hat{y}_i)$ and (x, y_i) be a global coordinate of D, local coordinates of $F=p^{-1}(0)$ and local coordinates of X. X is diffeomorphic to $D\times F$. Let $\sigma: D\times F\to X$ be a diffeomorphism. Then y_i is a differentiable function in x and $\hat{y}_i$. Moreover, we can choose σ such that y_i depends holomorphically on x (see Kuranishi [3]). By $\hat{u}_i(x, \hat{y}_i)$ and $\hat{w}_{ij}(x, \hat{y}_i)$ we denote, respectively, $\tilde{u}_i$ and w_{ij} considered as functions in $(x, \hat{y}_i)$. Now we define a differentiable function

$$u_i(x, y_i)=\int_C \hat{u}_i(x-\varepsilon\lambda(x)\xi, \hat{y}_i)\lambda(\xi)\,dv(\xi).$$

Then, for small $\varepsilon>0$, $(\partial^2 u_i/\partial y_i^\alpha \partial \bar{y}_i^\beta)$ is positive definite, because $\tilde{u}_i$ is differentiable in y_i. On the other hand, since w_{ij} is harmonic in (x, y_i) and y_i is holomorphic in x, $\hat{w}_{ij}$ is harmonic in x. Hence

$$\begin{aligned} u_i(x, y_i)&-u_j(x, y_i)\\ &=\int_C \hat{w}_{ij}(x-\varepsilon\lambda(x)\xi, \hat{y}_i)\lambda(\xi)\,dv(\xi)\\ &=\hat{w}_{ij}(x, \hat{y}_i)\\ &=w_{ij}(x, y_i). \end{aligned}$$

Thus $\{\sqrt{-1}\partial\bar{\partial}u_i\}$ defines a global d-closed $(1,1)$-form on X. By adding a suitable d-closed $(1,1)$-form on D, we can construct a Kähler metric ω on X, which satisfies our requirements.

4. Examples. Let $\hat{X}\to X$ be a monoidal transformation whose center is a point. Then we have the following

Proposition C. *X is Kähler (or projective) if and only if $\hat{X}$ is Kähler (or projective).*

In a forthcoming note [4], we shall prove the following theorem with the aid of the above Propositions A and B:

Theorem. *An elliptic surface admits a Kähler metric if and only if its first Betti numbers is even.*

References

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