143. *Exact Solution of a Certain Semi-Linear System of Partial Differential Equations related to a Migrating Predation Problem*

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1. Introduction. This paper is concerned with the solution of the initial value problem for the system of equations for $u_1(x, t)$ and $u_2(x, t)$:

$$L_i[u_i] \equiv \left(\frac{\partial}{\partial t} + c_i \frac{\partial}{\partial x}\right) u_i = \lambda_i u_1 u_2, \qquad (i=1,2) \tag{1.1}$$

with the bounded and measurable initial data

$$u_i(x, 0) = u_i^0(x), |x| < \infty. \tag{1.2}$$

The system (1.1) is the simplest hyperbolic one describing the non-linear coupling (characterized by parameters λ_1 and λ_2) between two waves propagating along the x-axis with constant velocities c_1 and c_2 respectively. If we put $c_1 = \lambda_1 = 1$ and $c_2 = \lambda_2 = -1$ it is reduced to the system proposed by Yamaguti [1] in order to describe a time history of the distribution of predator $u_1(t, x)$ and prey $u_2(t, x)$ running on a straight line in the opposite directions. Yamaguti [1] and Yoshikawa and Yamaguti [2] have given extensive studies of this system and have derived many important asymptotic properties of solutions as $t \to \infty$ without solving the equations explicitly. As far as the author is aware no explicit solution of our problem is found in the literature, in spite of the fact that it is reducible to the form amenable to Moutard's theorem [3].

The aim of this paper is to give the explicit solution of our problem and its version by means of a transformation analogous to that used by Hopf [4] and Cole [5] in their derivation of the solution of the Burgers equation. Several illustrating examples substantiating Yamaguti and Yoshikawa's prediction are given.

2. General solution. The solution u_i of (1.1) is derivable from the function ϕ:

$$u_i = \lambda_j^{-1} L_j[\phi], \qquad (j \neq i) = 1 \text{ or } 2 \tag{2.1}$$

provided that ϕ satisfies the equation

$$L_1 L_2[\phi] = L_1[\phi] L_2[\phi]. \tag{2.2}$$

Here and hereafter the suffices i and j denote the pair 1 and 2 or 2 and 1.

Let us introduce the new function Φ defined by

(2.3) $$\phi = -\log \Phi.$$

Then, (2.2) yields

(2.4) $$\begin{aligned} -L_1L_2[\phi] &= L_1L_2[\log \Phi] = L_1L_2[\Phi]/\Phi - L_1[\Phi]L_2[\Phi]/\Phi^2 \\ &= -L_1[\log \Phi]L_2[\log \Phi] = -L_1[\Phi]L_2[\Phi]/\Phi^2, \end{aligned}$$

which shows that Φ is given by the general solution of the linear equation

(2.5) $$L_1L_2[\Phi]=0,$$

i.e.

(2.6) $$\Phi = F_1(X_1) + F_2(X_2),$$

where

(2.7) $$X_i = x - c_i t,$$

and F_i ($i=1,2$) are arbitrary functions to be determined from e.g. the initial conditions.

Introducing (2.3) with (2.6) into (2.1) we have

(2.8) $$u_i = -\frac{1}{\lambda_j}\frac{1}{\Phi}L_j[\Phi] = \frac{1}{\lambda_j}(c_i - c_j)\frac{1}{\Phi}F_i'(X_i),$$

which is easily verified to satisfy (1.1) if we note $L_i[F_i(X_i)]=0$.

3. Initial value problem. In order to satisfy the initial conditions (1.2) we have to determine two arbitrary functions in (2.6) or (2.8) from the two equations

(3.1) $$\frac{F_i'(x)}{\Phi(x,0)} = \frac{F_i'(x)}{F_1(x)+F_2(x)} = g_i(x) \equiv \frac{\lambda_j}{c_i - c_j}u_i^0(x).$$

Equations (3.1) are solved by quadratures to give

(3.2) $$W(x) \equiv F_1(x) + F_2(x) = \exp\left\{\int_0^x [g_1(\xi)+g_2(\xi)]d\xi\right\}$$

and

(3.3) $$F_i(x) = \int_0^x W(\xi)g_i(\xi)d\xi + F_i(0),$$

where we have normalized F_i so that $W(0)=1$.

Introducing these expressions into (2.8) we have

(3.4) $$u_i(x,t) = W(X_i)u_i^0(X_i)\Big/\left[1+\sum_{i=1}^{2}\int_0^{X_i} W(\xi)g_i(\xi)d\xi\right]$$

which is proved to satisfy the initial conditions (1.2) if we use the identity derived from (3.2):

(3.5) $$[g_1(\xi)+g_2(\xi)]W(\xi) = W'(\xi).$$

Further reduction of (3.4) by the use of (3.5) in the denominator yields the amplification factor

(3.6) $$A_i(x,t) \equiv u_i(x,t)/u_i^0(X_i) = \left[1+\int_{X_i}^{X_j} g_j(\xi)W(\xi)d\xi/W(X_i)\right]^{-1}$$

or

(3.7) $$A_i(x,t) = [W(X_i)/W(X_j)]\Big/\left[1+\int_{X_j}^{X_i} g_i(\xi)W(\xi)d\xi/W(X_j)\right].$$

Comparing (3.6) for $i=1$ with (3.7) for $i=2$, we obtain the simple relation

$$(3.8) \qquad A_2(x,t)/A_1(x,t)=W(X_2)/W(X_1)=\exp\int_{X_1}^{X_2}(g_1+g_2)d\xi.$$

These are final forms of our exact solution of the initial value problem.

4. Special cases. 1) *Amplification at the invading front.* Let us assume

$$(4.1) \qquad c_1-c_2=c>0$$

and consider the value of u_1 at

$$(4.2) \qquad x=c_1t \quad \text{i.e. } X_1=x-c_1t=0$$

for the initial value

$$(4.3) \qquad u_1^0(x)=0 \quad \text{i.e. } g_1(x)=0 \quad \text{for } x>0.$$

Then, (3.6) for $i=1$ yields

$$(4.4) \qquad A_1=u_1(ct,t)/u_1^0(0)=1\Big/\left\{1+\int_0^{X_2}g_2(\xi)\left[\exp\int_0^{\xi}g_2d\xi\right]d\xi\right\},$$

where

$$(4.5) \qquad X_2=x-c_2t=ct>0.$$

Equation (4.4) is integrated to give

$$(4.6) \qquad A_1=\exp\int_0^{ct}[-g_2(\xi)]d\xi=\exp\left[\frac{1}{c}\int_0^{ct}\lambda_1u_2^0(\xi)d\xi\right],$$

which shows the possibility of infinite growth of u_1, at and behind the front if the integral is positive and infinitely large as $t\to\infty$.

2) *Collision at* $t=0$. Let us specialize the initial value in 1) by additional assumption

$$(4.7) \qquad u_2^0(x)=0 \quad \text{i.e. } g_2(x)=0 \quad \text{for } x<0.$$

Then, it is evident from (3.4), (4.3) and (4.7) that

$$(4.8) \qquad u_1=0, \quad u_2=u_2^0(X_2) \quad \text{for } X_1>0$$

and

$$u_1=u_1^0(X_1), \quad u_2=0 \quad \text{for } X_2<0.$$

In the region of interaction i.e.

$$(4.9) \qquad c_2t<x<c_1t \quad \text{i.e. } X_1<0 \text{ and } X_2>0,$$

we have

$$(4.10) \qquad W(\xi)=W_i(\xi)\equiv\exp\int_0^{\xi}g_i(\xi)d\xi, \qquad i=\begin{cases}1 & \xi<0\\ 2 & \xi>0\end{cases}$$

and

$$(4.11) \qquad \int_{X_i}^{X_j}g_jWd\xi=\int_0^{X_j}g_jW_jd\xi=W(X_j)-1.$$

Therefore, (3.6) is reduced to

$$(4.12) \qquad A_i=W_i(X_i)/[W_1(X_1)+W_2(X_2)-1].$$

At the front $X_1=0$ and the rear front $X_2=0$ we have

$$(4.13) \qquad A_i=[W_j((-1)^jct)]^{-1} \quad \text{and} \quad A_j=1 \quad \text{at} \quad X_i=0.$$

Differentiating A_i given by (4.12) with respect to x and using (4.10) and (4.11) we have

(4.14) $A_i^{-2}\partial A_i/\partial x=\{-g_i(X_i)[1-W_j(X_j)]-g_j(X_j)W_j(X_j)\}/W_i(X_i)$.

3) *The predator and prey problem.* When g_i $(i=1,2)$ are non-positive, $W(\xi)$ is a decreasing positive function of ξ. In this case A_2 and A_1 are proved to be non-negative according to (3.6) and (3.8) if $X_2>X_1$, i.e. $c_1-c_2>0$.

Especially when u_i^0 $(i=1,2)$ are non-negative and $\lambda_1>0$, $\lambda_2<0$, u_i are non-negative and may be regarded as the population of predator and pray running on a straight line with velocities c_1 and c_2 respectively.

Various asymptotic behaviours of u_1 and u_2 as $t\to\infty$ have been predicted by Yamaguti and Yoshikawa for $c_1=\lambda_1=1$ and $c_2=\lambda_2=-1$ by use of comparison theorems without use of explicit solutions. On the assumptions $g_i\leqq 0$ and $c>0$, some of their important results may be summarized as follows

i) If g_1 and g_2 are bounded and g_1 is bounded away from zero i.e. $-M_1<g_1<-\delta<0$ and $-M_2<g_2\leqq 0$, u_1 is bounded and u_2 tends to zero.

ii) If $g_1=0$ for $x>0$ and $g_2(x)\notin L^1(0,\infty)$, $|u_1|$ increases infinitely behind the front $x\lesssim c_1t$.

iii) If g_1 and g_2 are periodic functions of x with the same wave length $l>0$, u_1 is periodic with respect to t as $t\to\infty$.

Proof of i). Let us write (3.7) for $i=1$ as

$$\frac{1}{A_1}=\frac{W(X_2)}{W(X_1)}+\int_{X_1}^{X_2}\frac{W(\xi)}{W(X_1)}|g_1(\xi)|\,d\xi \tag{4.15}$$

and note $X_2-X_1=ct>0$ as well as

$$\exp[-M(\xi-X_1)]<W(\xi)/W(X_1)<\exp[-\delta(\xi-X_1)]$$

for $X_1\leqq\xi\leqq X_2$, where $M=M_1+M_2$.

Then, we have

$$e^{-Mct}+(\delta/M)(1-e^{-Mct})<A_1^{-1}<e^{-\delta ct}+(M_1/\delta)[1-e^{-\delta ct}] \tag{4.16}$$

i.e.

$$\frac{M}{\delta+(M-\delta)e^{-Mct}}>A_1>\frac{\delta}{M_1-(M_1-\delta)e^{-\delta ct}}$$

and from (3.8)

$$A_1e^{-\delta ct}>A_2>A_1e^{-Mct}. \tag{4.17}$$

The estimations (4.16) and (4.17) prove our assertion.

Proof of ii). We have only to note the unbounded growth of A_1 given by (4.6) and the continuity of A_1 as we recede from $x=c_1t$ i.e. $X_1=0$ and $X_2=ct$.

As (4.14) shows that

$$\partial A_1/\partial x=\{-g_1[1-W_2(ct)]-g_2W_2(ct)\}>0$$

at the front, the amplification is maximum there.

Proof of iii). It is evident from the translational invariance of

(1.1) that u_1 and u_2 are periodic with respect to x.

Let us keep $X_1=x-c_1t$ finite and consider the limit of (4.15) as $t\to\infty$, so that $X_2=X_1+ct\to\infty$. Then $W(\xi)$ is exponentially small as $\xi\to\infty$ since $\int_\xi^{\xi+l} g_j(\xi)d\xi=-a_j<0$ are bounded away from zero. Therefore, (4.15) yields

$$\lim_{t\to\infty} A_1^{-1}=A_{1\infty}^{-1}\equiv\int_{X_1}^\infty |g_1(\xi)| \exp\left[\int_{X_1}^\xi (g_1+g_2)d\xi\right]d\xi<\infty. \tag{4.18}$$

If we make use of the periodicity of $u_{1\infty}=A_{1\infty}(X_1)u_1^0(X_1)$ with respect to x and its dependence only on $X_1=x-c_1t$ it is evident that u_1 is periodic with respect to t; the period being l/c_1.

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