140. *Double Centralizers of Torsionless Modules**)

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In this note, we make the assumption that a ring has an identity element and modules are unital. For a left R-module ${}_RM$ where R is a ring, $D=\mathrm{End}_R({}_RM)$ is an R-endomorphism ring of ${}_RM$ operating on the side opposite to the scalars. Then ${}_RM$ is considered as an (R, D)-bimodule. A D-endomorphism ring $Q=\mathrm{End}_D(M_D)$ of M_D is called a double centralizer of ${}_RM$.

Definition. Let ${}_RM$ and ${}_RU$ be left R-modules, ${}_RM$ is said to be ${}_RU$-torsionless in case for each non-zero element m of ${}_RM$, there exists an R-homomorphism ϕ of ${}_RM$ into ${}_RU$ such that $(m)\phi\neq 0$.

We say that a left R-module ${}_RM$ is torsionless if ${}_RM$ is ${}_RR$-torsionless and ${}_RN$ is faithful if ${}_RR$ is ${}_RN$-torsionless. Let Q be a double centralizer of a faithful left R-module ${}_RM$, then there exists a canonical ring monomorphism of R into Q, written as $R\subsetneq Q$. A faithful left R-module ${}_RM$ is said to have the double centralizer property if $R=Q$, where Q is a double centralizer of ${}_RM$.

Definition. A ring R is left QF-1 if every faithful left R-module has the double centralizer property.

QF-1 rings were first described by R. M. Thrall (1948 [4]) and have been examined by many authors. It was proved that the double centralizer of a faithful torsionless left R-module is a rational extension of R_R. Furthermore the double centralizer of a dominant left R-module is a maximal right quotient ring of R (see T. Kato [1] and H. Tachikawa [3]). In the section 1, the next theorem is proved.

Theorem. *Let R be a ring with minimum condition and U be the intersection of all left faithful two-sided ideals of R. Then U is also a left faithful two-sided ideal of R and the double centralizer of ${}_RU$ is a maximal right quotient ring of R.*

In the section 2, we shall prove that for a given faithful left R-module ${}_RM$, ${}_RM$ has the double centralizer property if and only if ${}_KKe$ has the double centralizer property, where

$$K=\begin{pmatrix} R & M \\ \mathrm{Hom}_R({}_RM, {}_RR) & \mathrm{End}_R({}_RM)\end{pmatrix} \quad \text{and} \quad e=\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\in K.$$

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1. We shall first prove the next theorem which is similar to K. Morita's result [2, Theorem 1.1.].

Theorem 1. *Let $_RM$ and $_RU$ be left R-modules. If the following conditions are satisfied:*

(1) *There exists the following R-exact sequence:*

$$\oplus\, {}_RM \to {}_RU \to 0,$$

(2) *If $\sum m_i\phi_i = 0$, $m_i \in {}_RM$, $\phi_i \in \mathrm{Hom}_R({}_RM, {}_RU)$, then $\sum (qm_i)\phi_i = 0$ for any $q \in Q$,*

(3) *For each non-zero element q of Q, there exist $m \in {}_RM$ and $\phi \in \mathrm{Hom}_R({}_RM, {}_RU)$ such that $(qm)\phi \neq 0$,*

then we have $Q \subsetneq \bar{Q}$ where Q and $\bar{Q}$ are double centralizers of $_RM$ and $_RU$ respectively.

Proof. For any $q \in Q$, we define $\bar{q}$ as $\bar{q}(\sum m_i\phi_i) = \sum (qm_i)\phi_i$. Then the mapping: $q \to \bar{q}$ is well-defined by (2). An element $\bar{q}$ is contained in $\bar{Q}$ since

$$\bar{q}((\sum m_i\phi_i)d) = \bar{q}(\sum m_i\phi_i d) = \sum (qm_i)\phi_i d = (\sum (qm_i)\phi_i)d = (\bar{q}(\sum m_i\phi_i))d$$

for any $d \in \mathrm{End}_R({}_RU)$. And this mapping: $q \to \bar{q}$ is a ring monomorphism of Q into $\bar{Q}$ by (3).

Lemma 2. *If $_RU$ is $_RM$-torsionless, then the condition (2) of Theorem 1 is satisfied.*

Proof (c.f. T. Kato [1]). If $\sum (qm_i)\phi_i \neq 0$, $q \in Q$, $m_i \in {}_RM$, $\phi_i \in \mathrm{Hom}_R({}_RM, {}_RU)$, then there exists $d \in \mathrm{Hom}_R({}_RU, {}_RM)$ such that $(\sum (qm_i)\phi_i)d \neq 0$ since $_RU$ is $_RM$-torsionless. And

$$(\sum (qm_i)\phi_i)d = \sum (qm_i)\phi_i d = \sum q(m_i\phi_i d)$$

by $q \in Q = \mathrm{Hom}_D(M_D, M_D)$ and $\phi_i d \in D = \mathrm{Hom}_R({}_RM, {}_RM)$. Further

$$\sum q(m_i\phi_i d) = q(\sum (m_i\phi_i)d) = q((\sum m_i\phi_i)d) \neq 0.$$

Then we have $\sum m_i\phi_i \neq 0$.

Since the condition (3) of Theorem 1 is satisfied if $_RM$ is $_RU$-torsionless, we have the following.

Lemma 3. *Let $_RM$ and $_RU$ be left R-modules. If the following conditions are satisfied:*

(1) *There exists the following R-exact sequence:*

$$\oplus\, {}_RM \to {}_RU \to 0,$$

(2) *$_RU$ is $_RM$-torsionless,*

(3) *$_RM$ is $_RU$-torsionless,*

then we have $Q \subsetneq \bar{Q}$ where Q and $\bar{Q}$ are double centralizers of $_RM$ and $_RU$ respectively.

Lemma 4. *Let A and B be left faithful two-sided ideals of a ring R. Then $A \cap B$ is also a left faithful two-sided ideal of R.*

Proof. Clearly $AB = \{\sum a_ib_i \mid a_i \in A, b_i \in B\}$ is a two-sided ideal contained in a two-sided ideal $A \cap B$. For each non-zero element r of R, there exists $a \in A$ such that $ra \neq 0$ since A is left faithful. Similarly for $ra \neq 0$, there exists $b \in B$ such that $(ra)b \neq 0$ since B is left faithful. By $r(ab) \neq 0$, $ab \in AB$, AB is left faithful. Hence $A \cap B$ is also left faithful.

Definition. For a left R-module ${}_RM$, the sum of all R-homomorphic images of ${}_RM$ into ${}_RR$ is called a trace ideal of ${}_RM$, written as $\mathrm{Tr}\,({}_RM)$.

Theorem 5. *Let R be a ring with minimum condition and U be the intersection of all left faithful two-sided ideals of R. Then U is also a left faithful two-sided ideal of R and the double centralizer of ${}_RU$ is a maximal right quotient ring of R.*

Proof. If R is a ring with minimum condition, then U is a left faithful two-sided ideal of R because of Lemma 4. For any faithful torsionless left R-module ${}_RM$, let $\mathrm{Tr}\,({}_RM)$ be a trace ideal of ${}_RM$ and Q, Q' be double centralizers of ${}_RM, {}_R\mathrm{Tr}\,({}_RM)$ respectively. By Lemma 3, we have $Q \subsetneq Q'$. Since $\mathrm{Tr}\,({}_RM)U$ is also a left faithful two-sided ideal contained in U, then $\mathrm{Tr}\,({}_RM)U = U$. In this case, Q' is contained in the double centralizer $\overline{Q}$ of ${}_RU$ by Lemma 3. Thus we have $Q \subsetneq \overline{Q}$. This ring $\overline{Q}$ is a maximal right quotient ring of R since R has a dominant left R-module (see T. Kato [1]).

Theorem 6. *Let R be a left cogenerator ring. Then the following statements are equivalent:*

(1) *R is a left QF-1 ring.*

(2) *Every faithful left ideal of R has the double centralizer property.*

(3) *Every left faithful two-sided ideal of R has the double centralizer property.*

(4) *Every left faithful trace ideal of R has the double centralizer property.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) is clear. (4)$\Rightarrow$(1). For any faithful left R-module ${}_RM$, let $\mathrm{Tr}\,({}_RM)$ be a trace ideal of ${}_RM$ and Q, $\overline{Q}$ be double centralizers of ${}_RM, {}_R\mathrm{Tr}\,({}_RM)$ respectively. By Lemma 3, we have $Q \subsetneq \overline{Q}$ and (4) implies $Q = R$.

2. In this section, let ${}_RM$ be a faithful left R-module, $D = \mathrm{End}_R\,({}_RM)$ and $Q = \mathrm{End}_D\,(M_D)$. It is easily shown that the canonical mapping

$$\eta : \mathrm{Hom}_R\,({}_RM, {}_RR) \to \mathrm{Hom}_D\,(M_D, D_D)$$

is a (D, R)-monomorphism and the canonical mapping

$$\rho : \mathrm{Hom}_D\,(M_D, D_D) \to \mathrm{Hom}_Q\,({}_QM, {}_QQ)$$

is a (D, Q)-isomorphism. We define a ring K as

$$K = \begin{pmatrix} R & M \\ \mathrm{Hom}_R\,({}_RM, {}_RR) & D \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} r & m \\ \phi & d \end{pmatrix} \mid r \in R, m \in M, \phi \in \mathrm{Hom}_R\,({}_RM, {}_RR), d \in D \right\}.$$

In this ring K, for $m \in M$ and $\phi \in \mathrm{Hom}_R({}_RM, {}_RR)$, let $m\phi \in R$ be as usual but ϕm means $(\eta\phi)m \in D$.

Lemma 7. *Let ${}_RM$, D, Q and K be as above. If ${}_RM$ is faithful, then ${}_KKe$ is faithful, where*

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in K.$$

Lemma 8. *Let ${}_RM$, D, Q, K and $e \in K$ be as above. Then the double centralizer of ${}_KKe$ is a ring*

$$\begin{pmatrix} Q & M \\ \mathrm{Hom}_D(M_D, D_D) & D \end{pmatrix}.$$

Finally we describe our main theorem which is thought to be useful in solving later problems.

Theorem 9. *Let ${}_RM$, D, Q, K and $e \in K$ be as above. Then ${}_RM$ has the double centralizer property if and only if ${}_KKe$ has the double centralizer property.*

References

[1] T. Kato: *U*-dominant dimension and *U*-localization (unpublished).
[2] K. Morita: On algebras for which every faithful representation is its own second commutator. Math. Z., **69**, 429–434 (1958).
[3] H. Tachikawa: On left *QF*-3 rings. Pacific J. Math., **32**, 255–268 (1970).
[4] R. M. Thrall: Some generalizations of quasi-Frobenius algebras. Trans. Amer. Math. Soc., **64**, 173–183 (1948).